Zero forcing propagation time on oriented graphs

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Abstract

Zero forcing is an iterative coloring procedure on a graph that starts by
initially coloring vertices white and blue and then repeatedly applies the fol-
lowing rule: if any vertex colored blue has a unique (out-)neighbor that is
colored white, then that neighbor is changed from white to blue. Any initial
set of blue vertices which can change the entire graph to blue is called a zero
forcing set. In this paper we consider the minimum number of iterations needed
for this color change rule to color all of the vertices blue, also known as the
propagation time, for oriented graphs. We produce oriented graphs with both
high and low propagation times, consider the possible propagation times for the
orientations of a fixed graph, and look at balancing the size of a zero forcing
set and the propagation time.

Keywords. zero forcing process, propagation time, oriented graphs, Hessenberg
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1 Introduction

Given a directed graph with no loops (i.e., a simple digraph) there are many possible processes that can be used to simulate information spreading. In the simplest model we can have two states on each vertex, knowing or not knowing (which we represent using the colors blue and white, respectively), and then have simple threshold rules for when a vertex changes from not knowing to knowing (i.e., changes from white to blue). For each possible threshold rule we run into a variety of questions including finding the minimum number of vertices colored blue that will eventually change all the vertices blue, or finding the length of time it takes for a graph to become blue. The goal in this paper is to consider a particular threshold rule, known as zero forcing, and to focus on the amount of time it takes to turn all the vertices blue, known as propagation time, on oriented graphs.

The zero forcing process on a simple digraph is based on the following coloring rule: If a blue vertex has exactly one white out-neighbor, then that out-neighbor will change from white to blue. If we think of this in terms of rumor spreading then this can be rephrased in the following way: “If I know a secret and all except one of my friends knows the same secret, then I will share that secret with my friend that doesn’t know.”

The process of zero forcing was introduced originally for (undirected) graphs [2] and extended to digraphs in Barioli et al. [3]. Zero forcing for simple digraphs was studied in [9, 4]. This process is of interest because there is a relationship between the minimum number of vertices initially colored blue that can transform the entire graph to blue (also known as the zero forcing number), and the geometric multiplicity of the eigenvalue 0 for a matrix associated with a graph. In general, the zero forcing number can be determined computationally but is NP-hard [1]; it has been determined for several families of graphs (more information can be found in the recent survey by Fallat and Hogben [8]).

While most of the focus of the literature has been on the determination of the zero forcing number, another natural question to examine is the amount of time it takes to turn all of the vertices blue, i.e., the propagation time. The initiation of this study for undirected graphs was given in the paper of Hogben et al. [10] where extremal configurations were determined (i.e., graphs which propagate as quickly or as slowly as possible) and in Chilakamarri et al. [7] where propagation time (there called iteration index) was computed for some families of graphs. The goal of this paper is to expand the study of propagation time to oriented graphs. In particular, we will see that there are some subtle and important distinctions between undirected graphs and oriented graphs.

We will proceed as follows: In the remainder of the introduction we introduce the notation and give precise terminology. In Section 2 we show that the propagation time is not affected when the direction of all the arcs of an oriented graph are reversed. In Sections 3 and 4 we look at oriented graphs that have high and low propagation times.
respectively, and in Section 5 we consider orientation propagation intervals. Finally, in Section 6 we consider what happens when we balance the size of the zero forcing set with the propagation time; in particular we show that, unlike in simple graphs, we cannot always obtain significant savings.

1.1 Terminology and definitions

We use $G = (V(G), E(G))$ to denote a graph and $\Gamma = (V(\Gamma), E(\Gamma))$ to denote a digraph, where $V$ and $E$ are the respective vertex and edge (or arc) sets. For a digraph $\Gamma$ having $u, v \in V(\Gamma)$ and $(u, v) \in E(\Gamma)$, we say that $v$ is an out-neighbor of $u$ and that $u$ is an in-neighbor of $v$. The set of all in-neighbors of $v$ is denoted $N^-(v)$ and the cardinality of $N^-(v)$ is the in-degree of $v$, denoted $\deg^-(v)$. Similarly, the set of all out-neighbors of $v$ is $N^+(v)$ and the cardinality of $N^+(v)$ is the out-degree, denoted $\deg^+(v)$.

For a simple digraph $\Gamma$, the zero forcing process can be described as follows. Let $B \subseteq V(\Gamma)$, let $B(0) := B$ and iteratively define $B(t+1)$ as the set of vertices $v$ where for some $u \in \bigcup_{i=0}^{t} B(i)$ we have that $v$ is the unique out-neighbor of $u$ that is not in $\bigcup_{i=0}^{t} B(i)$. Here $B(0)$ represents the initial set of vertices colored blue, and at each stage we color as many vertices blue as possible (i.e., we apply the coloring rule simultaneously to all vertices). We say a set $B$ is a zero forcing set, or ZFS, if $\bigcup_{i=0}^{t} B(i) = V(\Gamma)$. Further, the propagation time of $B$, denoted $pt(\Gamma, B)$, is the minimum $t$ so that $\bigcup_{i=0}^{t} B(i) = V(\Gamma)$ (i.e., the minimum amount of time needed for $B$ to color the entire graph blue).

One way to achieve fast propagation is to simply let $B(0) = V(\Gamma)$, and be done at time 0. However, we are primarily interested in the propagation time of minimum sized $B$. In particular, for a simple digraph $\Gamma$ we will let $Z(\Gamma)$ denote the minimum size of a zero forcing set for $\Gamma$. We then define propagation time as follows:

$$pt(\Gamma) = \min_{\|B\| = Z(\Gamma)} \text{pt}(\Gamma, B)$$

In this paper we use $\overrightarrow{G}$ to denote an oriented graph (i.e., a simple digraph where there are no double arcs: if $(u, v)$ is an arc in $\overrightarrow{G}$ then $(v, u)$ is not an arc in $\overrightarrow{G}$).

Example 1.1. Consider the oriented graph $\overrightarrow{G}$ shown in Figure 1. Since the vertices $c, d$ and $f$ are not the out-neighbors of any vertices, they cannot be changed to blue by the coloring rule. Therefore these three vertices must be in a zero forcing set of $\overrightarrow{G}$.

We now show that these three vertices form a zero forcing set (and in particular this is the unique minimum cardinality zero forcing set), allowing us to conclude $Z(\overrightarrow{G}) = 3$.

So now suppose that $B(0) = \{c, d, f\}$, and we mark these vertices by coloring them blue (see $t = 0$ in Figure 1). Since $e$ is the unique white out-neighbor of $f$ then we can color $e$ blue, but this is the only vertex which can be colored and so we have $B(1) = \{e\}$. The state of our coloring is now shown in $t = 1$ in Figure 1. At this
point we note that $b$ is the unique white out-neighbor of $e$ and that $a$ is the unique white out-neighbor of $d$ and so we can color both of them and we have $B(2) = \{a, b\}$.

At this stage all the vertices are blue and so we can conclude that the propagation time corresponding to this set is 2, but, as already noted, this is the unique minimum zero forcing set and so we have $pt(\overrightarrow{G}) = 2$.

![Figure 1: An example of propagation for the zero forcing process](image)

When a white vertex $v$ is the unique white out-neighbor of a blue vertex $u$, then we say that $u$ forces $v$ to change its color, and we write $u \rightarrow v$. Given a set $B$ we can consider the union of arcs that correspond to forces that were used in coloring the graph. This collection of arcs is known as a set of forces, and denoted by $\mathcal{F}$. When there is a white vertex that could be changed to blue by two different in-neighbors we put only one of the corresponding arcs in $\mathcal{F}$. In particular, for a given set $B$ there are possibly many different forcing sets. However, whether or not $B$ is a zero forcing set, and similarly the propagation time, is not dependent on which choices we make in including arcs (see [5] and [10] for more information).

The subdigraph found by considering the arcs from $\mathcal{F}$ corresponds to a collection of disjoint directed paths, where each vertex in $B$ is the tail of a path. In particular, at each iteration in computing propagation we would have added at most one vertex on each path and so we have $|B(t)| \leq |B|$. An immediate consequence is the following observation.

**Observation 1.2.** For any simple digraph $\Gamma$,

$$\frac{|\Gamma| - Z(\Gamma)}{Z(\Gamma)} \leq pt(\Gamma) \leq |\Gamma| - Z(\Gamma).$$

For a simple graph $G$, we let $\overrightarrow{G}$ denote an orientation of $G$, i.e., $\overrightarrow{G}$ is an oriented graph where ignoring the orientations of the arcs gives the graph $G$. For a given graph $G$ there are many possible orientations and this gives rise to the following problem: For a given graph $G$, find $\{pt(\overrightarrow{G})\}$ as $\overrightarrow{G}$ ranges over all possible orientations of $G$.

Finally, we note that without loss of generality we can assume that our graphs are (weakly) connected. This is because once we know the zero forcing numbers...
and propagation times on each component, we know the zero forcing number and propagation time on the whole graph. This is given in the following observation.

**Observation 1.3.** For any simple digraph $\Gamma$ with connected components $\Gamma_1, \ldots, \Gamma_h$,

$$Z(\Gamma) = \sum_{i=1}^{h} Z(\Gamma_i) \text{ and } pt(\Gamma) = \max_{i=1}^{h} pt(\Gamma_i).$$

### 2 Reversing Arcs

Although our focus is on oriented graphs, the results in this section are true for simple digraphs, so we state them that way. Given a simple digraph $\Gamma$, we let $\Gamma^T$ be the simple digraph where the direction of each arc has been reversed. (Note that the adjacency matrix of $\Gamma^T$ is the transpose of the adjacency matrix of $\Gamma$, which motivates the notation.) Reversing the arcs will generally change what the zero forcing sets are and how they propagate. However, we show in Theorem 2.5 below that $pt(\Gamma^T) = pt(\Gamma)$, following the arguments in [10].

Let $\Gamma$ be a simple digraph, $B$ a minimum zero forcing set of $\Gamma$, and $F$ a set of forces of $B$. The **terminus** of $F$, denoted $\text{Term}(F)$, is the set of vertices that do not perform a force in $F$, i.e., these are the heads of the directed paths formed by $F$ (note that if a vertex in $B$ never forces then it is both the tail and head on a path of length 0). Let $\text{Rev}(F)$ correspond to the set found by reversing the direction of each arc. Note that $F \subseteq E(\Gamma)$ and $\text{Rev}(F) \subseteq E(\Gamma^T)$.

**Proposition 2.1** ([5]). Let $\Gamma$ be a simple digraph, $B$ a minimum zero forcing set of $\Gamma$, and $F$ a set of forces of $B$. Then $\text{Term}(F)$ is a zero forcing set for $\Gamma^T$ and $\text{Rev}(F)$ is a set of forces. Hence $Z(\Gamma^T) = Z(\Gamma)$.

We previously have defined propagation in terms of an initial set $B$, but we can also define propagation using the the set of forces $F$.

**Definition 2.2.** Let $\Gamma = (V, E)$ be a simple digraph and $B$ a zero forcing set of $\Gamma$. For a set of forces $F$ of $B$ that colors all vertices, define $F^{(0)} = B$ and for $t \geq 0$, let $F^{(t+1)}$ be the set of vertices $w$ such that, for some $v \in \bigcup_{i=0}^{t} F^{(i)}$, the arc $v \rightarrow w$ appears in $F$, $w \notin \bigcup_{i=0}^{t} F^{(i)}$, and $w$ is the only neighbor of $v$ not in $\bigcup_{i=0}^{t} F^{(i)}$. (Note that the set $F$ is a collection of arcs while the sets $F^{(i)}$ are collections of vertices.) The **propagation time of $F$ in $\Gamma$**, denoted $pt(\Gamma, F)$, is the minimum $t$ such that $\bigcup_{i=0}^{t} F^{(i)} = V(\Gamma)$.

We now give a connection between the propagation time given by $F$ in $\Gamma$ and the propagation time given by $\text{Rev}(F)$ in $\Gamma^T$.

**Lemma 2.3.** Let $\Gamma = (V, E)$ be a simple digraph, $B$ a minimum zero forcing set, $F$ a set of forces of $B$, and $1 \leq t \leq pt(\Gamma)$. If $v \rightarrow u \in F$ such that $u \in F^{(pt(\Gamma)-t+1)}$, then $v \in \bigcup_{i=0}^{t} \text{Rev}(F)^{(i)}$. 5
Proof. By Proposition 2.1 we have $\text{Term}(\mathcal{F})$ is a zero forcing set for $\Gamma^T$ with forcing set $\text{Rev}(\mathcal{F})$. We establish the result by induction on $t$. For $t = 1$, let $u \in \mathcal{F}_{\text{pt}(\Gamma)}$. Then $u \in \text{Term}(\mathcal{F}) = \text{Rev}(\mathcal{F})^{(0)}$. If $x \neq v$ is an in-neighbor of $u$ in $\mathcal{F}$, then $x$ cannot force in $\mathcal{F}$ since $u \in \mathcal{F}_{\text{pt}(\Gamma)}$. So $x \in \text{Term}(\mathcal{F}) = \text{Rev}(\mathcal{F})^{(0)}$. Hence, $v$ is the only white out-neighbor of $u$ in $\text{Rev}(\mathcal{F})$. So $v \in \text{Rev}(\mathcal{F})^{(1)}$.

Assume that the claim is true for $1 \leq s \leq t$. Suppose $u \in \mathcal{F}_{\text{pt}(\Gamma)-(t+1)+1}$. Then $v \rightarrow u$ at time $\text{pt}(\Gamma) - t$, so $u$ cannot perform a force in $\mathcal{F}$ until $\text{pt}(\Gamma) - t + 1$ or later. Thus $u \in \bigcup_{i=0}^{t} \text{Rev}(\mathcal{F})^{(i)}$ by definition. If $x \neq v$ is an in-neighbor of $u$ in $\mathcal{F}$, then $x$ cannot perform a force in $\mathcal{F}$ until $\text{pt}(\Gamma) - t + 1$ or later. So $x \in \bigcup_{i=0}^{t} \text{Rev}(\mathcal{F})^{(i)}$. Thus, if $v \notin \bigcup_{i=0}^{t} \text{Rev}(\mathcal{F})^{(i)}$, then $v \in \text{Rev}(\mathcal{F})^{(t+1)}$, i.e., $v \in \bigcup_{i=0}^{t+1} \text{Rev}(\mathcal{F})^{(i)}$ as desired. □

Corollary 2.4. Let $\Gamma = (V, E)$ be a simple digraph, $B$ a minimum zero forcing set of $\Gamma$, and $\mathcal{F}$ a forcing set of $B$. Then $\text{pt}(\Gamma^T, \text{Rev}(\mathcal{F})) \leq \text{pt}(\Gamma, \mathcal{F})$.

A minimum zero forcing set $B$ of $\Gamma$ is said to be an efficient zero forcing set if $\text{pt}(\Gamma, B) = \text{pt}(\Gamma)$. A set of forces $\mathcal{F}$ of an efficient forcing set $B$ is efficient if $\text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma)$.

Theorem 2.5. Let $\Gamma = (V, E)$ be a simple digraph. Then $\text{pt}(\Gamma^T) = \text{pt}(\Gamma)$.

Proof. Choose an efficient zero forcing set $B$ and efficient set of forces $\mathcal{F}$, so $\text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma)$. Then by Corollary 2.4, $\text{pt}(\Gamma^T) \leq \text{pt}(\Gamma^T, \text{Rev}(\mathcal{F})) \leq \text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma)$. By reversing the roles of $\Gamma$ and $\Gamma^T$, we obtain the reverse inequality, and in particular can conclude that we have equality. □

3 High Propagation Times

In this section we focus on oriented graphs that have high propagation times. The two key elements to obtain high propagation time are a small zero forcing set and few simultaneous forces occurring at each step. Although our primary interest is oriented graphs, much of the literature deals with simple digraphs.

A Hessenberg path on $n$ vertices is a simple digraph on the vertices $v_1, v_2, \ldots, v_n$ that contains the sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots (v_{n-1}, v_n)$, and does not contain any arc of the form $(v_i, v_j)$ with $j > i + 1$. (Note that no restrictions are placed on arcs of the form $(v_i, v_j)$ with $i > j + 1$, i.e., “back” arcs are allowed.)

Observation 3.1. [9] For any simple digraph $\Gamma$, $Z(\Gamma) = 1$ if and only if $\Gamma$ is a Hessenberg path. (This includes a single isolated vertex.)

Combining this with Observations 1.2 and 1.3 we have the following.

Observation 3.2. For any simple digraph $\Gamma$, the following are equivalent:

1. $Z(\Gamma) = 1$. 
2. $\text{pt}(\Gamma) = |\Gamma| - 1$.

3. $\Gamma$ is a Hessenberg path.

A simple digraph $\Gamma$ is a \textit{digraph of two parallel Hessenberg paths} if $\Gamma$ is not itself a Hessenberg path, and $V(\Gamma) = \{p_1, \ldots, p_s, q_1, \ldots, q_t\}$ (where $s, t \neq 0$), $\Gamma[\{p_1, \ldots, p_s\}]$ and $\Gamma[\{q_1, \ldots, q_t\}]$ are Hessenberg paths, and there do not exist $i, j, k, \ell$ such that $i < j$, $k < \ell$, $(p_k, q_j) \in E(\Gamma)$, and $(q_i, p_\ell) \in E(\Gamma)$ \cite{5}. In other words, there are no forward crossing arcs between the two Hessenberg paths. An \textit{oriented graph of two parallel Hessenberg paths} is a digraph of two parallel Hessenberg paths with no double arcs.

\textbf{Theorem 3.3.} \cite{5} For any simple digraph $\Gamma$, $Z(\Gamma) = 2$ if and only if $\Gamma$ is a digraph of two parallel Hessenberg paths.

Resuming our focus on oriented graphs, the next statement is now immediate from Observations 1.2 and 3.2 and Theorem 3.3.

\textbf{Observation 3.4.} For any oriented graph $\overrightarrow{G}$, if $\text{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 2$, then $Z(\overrightarrow{G}) = 2$ and $\overrightarrow{G}$ is an oriented graph of two parallel Hessenberg paths.

Note that the converse of Observation 3.4 is false, as shown in the next example.

\textbf{Example 3.5.} Let $\overrightarrow{P_4}$ be the oriented graph in Figure 2. Then $Z(\overrightarrow{P_4}) = 2$ because vertices $a$ and $b$ have in-degree zero (and $\overrightarrow{P_4}$ is an oriented graph of two parallel Hessenberg paths), but $\text{pt}(\overrightarrow{P_4}) = 1 \neq |\overrightarrow{P_4}| - 2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A counterexample to the converse of Observation 3.4}
\end{figure}

Our goal is to classify the orientations $\overrightarrow{G}$ of graphs $G$ for which $\text{pt}(\overrightarrow{G}) = |G| - 2$. We already know that $\text{pt}(\overrightarrow{G}) = |G| - 1$ if and only if $Z(\overrightarrow{G}) = 1$; that is, if and only if $\overrightarrow{G}$ is a Hessenberg path.

In order for $\text{pt}(\overrightarrow{G}) = |G| - 2$, it must then be the case that $Z(\overrightarrow{G}) = 2$, since if $Z(\overrightarrow{G}) \geq 3$, we are guaranteed that $\text{pt}(\overrightarrow{G}) < |G| - 2$ by Observation 1.2. Therefore, $\overrightarrow{G}$ must be an oriented graph of two parallel Hessenberg paths. That is, $V(\overrightarrow{G}) = \{p_1, \ldots, p_s, q_1, \ldots, q_t\}$ ($s, t \geq 1$) and we are able to express $\overrightarrow{G}$ as two Hessenberg paths, $P = \Gamma[\{p_1, p_2, \ldots, p_s\}]$ and $Q = \{\{q_1, q_2, \ldots, q_t\}\}$, with possible arcs between $P$ and $Q$ such that there are no pairs of arcs of the form $(p_i, q_\ell)$ and $(q_k, p_j)$ where
\(i < j\) and \(k < \ell\). If we can express \(\overrightarrow{G}\) as an oriented graph of two parallel Hessenberg paths \(P\) and \(Q\), we refer to that particular way of writing \(\overrightarrow{G}\) as \(\overrightarrow{G}(P, Q)\).

In this case, \(B = \{p_1, q_1\}\) is a minimum zero forcing set for \(\overrightarrow{G}\). Moreover, if \(B\) is any minimum zero forcing set for \(\overrightarrow{G}\), \(\overrightarrow{G}\) can be expressed as \(\overrightarrow{G}(P', Q')\) where \(B = \{p'_1, q'_1\}\). To achieve \(\text{pt}(\overrightarrow{G}) = |G| - 2\), it must be the case that \(\text{pt}(\overrightarrow{G}, B) = |G| - 2\) for all zero forcing sets \(B\) of order two. This motivates the following notation and definition which extend the definitions in [10] to oriented graphs.

Given \(\overrightarrow{G}(P, Q)\), we let \(V(P)\) and \(V(Q)\) denote the vertices of \(P\) and \(Q\), respectively. If \(u, v\) are vertices of \(P\) (respectively \(Q\)), we say that \(u \prec v\) if \(u = p_i\) and \(v = p_j\) for some \(i < j\) (respectively \(u = q_k\) and \(v = q_l\) for some \(k < \ell\)). Furthermore, for \(i > 1\), we say that \(p_{i-1} = \text{prev}(p_i), q_{i-1} = \text{prev}(q_i), \text{next}(p_{i-1}) = p_i,\) and \(\text{next}(q_{i-1}) = q_i\). That is, if \(v = \text{next}(u)\) or \(u = \text{prev}(v)\), then \(u\) and \(v\) are in the same Hessenberg path.

**Definition 3.6.** Suppose \(\overrightarrow{G}(P, Q)\) is an orientation of two parallel Hessenberg paths as described above. Then we say that \(\overrightarrow{G}(P, Q)\) is a **zig-zag orientation** if \(\overrightarrow{G}(P, Q)\) contains a directed path \((z_1, z_2, \ldots, z_r)\), \((r \geq 2)\) for which \(z_1 = q_1, z_i \in V(Q)\) for \(i\) odd, \(z_i \in V(P)\) for \(i\) even, and \(z_i \prec z_{i+2}\) for \(1 \leq i \leq r - 2\). Note that if a directed path \((z_1, z_2, \ldots, z_r)\) exists where \(z_i \in V(P)\) for \(i\) odd and \(z_i \in V(Q)\) for \(i\) even, we may switch the roles of \(P, Q\) so the definition is met.

If \(\overrightarrow{G}(P, Q)\) is a zig-zag orientation, then we denote

\[
\text{out}(z_i) = \begin{cases} 
\{p_j \in V(P) : (z_i, p_j) \in E(\overrightarrow{G})\} & \text{for } i \text{ odd}; \\
\{q_k \in V(Q) : (z_i, q_k) \in E(\overrightarrow{G})\} & \text{for } i \text{ even}.
\end{cases}
\]

That is, \(\text{out}(z_i)\) is the set of vertices in \(N^+(z_i)\) that are not in the same Hessenberg path as \(z_i\).

We are now interested in characterizing the orientations \(\overrightarrow{G}\) for which \(\text{pt}(\overrightarrow{G}) = |G| - 2\). In the case that \(G\) is disconnected, the only possible \(\overrightarrow{G}\) for which \(\text{pt}(\overrightarrow{G}) = |G| - 2\) are \(\overrightarrow{G} = K_1 \cup \overrightarrow{H}\) where \(\overrightarrow{H}\) is any Hessenberg path on 1 or more vertices. Thus, we will only consider connected graphs.

The following theorem characterizes when a zero forcing set \(B\) of size two achieves \(\text{pt}(\overrightarrow{G}, B) = |G| - 2\).

**Theorem 3.7.** Let \(\overrightarrow{G}\) be an orientation of a connected graph \(G\) for which \(Z(\overrightarrow{G}) = 2\). Then \(\text{pt}(\overrightarrow{G}, B) = |G| - 2\) if and only if \(\overrightarrow{G}\) can be written as a zig-zag orientation \(\overrightarrow{G}(P, Q)\) with the following properties:

1. \(B = \{p_1, q_1\}\).

2. For \(1 \leq i \leq r - 1\), if \(u \in \text{out}(z_i)\) then \(u \preceq z_{i+1}\).
3. One of the following must hold:

(a) \( z_{r-1} = p_s \) or \( z_{r-1} = q_t \), and either \( z_r = \text{next}(z_{r-2}) \) or \( \text{prev}(z_r) \in \text{out}(z_{r-1}) \).

(b) \( z_{r-1} \neq p_s, z_{r-1} \neq q_t, z_r = p_s \) or \( z_r = q_t \), and if \( u \in \text{out}(z_r) \) then \( u \preceq z_{r-1} \).

(c) \( q_1 = q_t \) and \( z_2 = p_2 \).

Proof. Suppose \( \overrightarrow{G} \) is an orientation of \( G \) and \( B = \{p_1, q_1\} \) is a minimum zero forcing set with \( \text{pt}(\overrightarrow{G}, B) = |G| - 2 \). Then suppose \( P = (p_1, p_2, \ldots, p_s) \) and \( Q = (q_1, q_2, \ldots, q_t) \) are the forcing chains for \( B \). By construction, we may express \( \overrightarrow{G} \) as \( \overrightarrow{G}(P, Q) \).

Since \( \text{pt}(\overrightarrow{G}, B) = |G| - 2 \), only one force may occur at each time step of the forcing process. Without loss of generality, suppose the first force is performed by \( p_1 \). Let \( z_1 = q_1 \). If \( z_1 \) is able to perform a force at the same time step as \( p_1 \), then it must be the case that \( p_2 \in \text{out}(z_1) \) and \( p_i \notin \text{out}(z_1) \) for \( i > 2 \) because at each time step, only one white vertex can become blue. In this case, we let \( z_2 = z_r = p_2 \), resulting in a zig-zag orientation satisfying Properties 1, 2, and 3(c). Suppose \( z_1 \) may not perform a force at the first time step. Then, if \( q_1 = q_t \), there is at least one \( i > 2 \) for which \( p_i \in \text{out}(z_1) \). If \( q_1 \neq q_t \), there is at least one \( i > 1 \) for which \( p_i \in \text{out}(z_1) \). We consider the largest such \( i \) and let \( z_2 = p_i \).

Thus, \( z_1 \) may not perform a force until \( z_2 \) is blue. Forcing occurs along the path \( P \) until \( z_2 \) is blue. At this point, \( z_1 \) may force \( q_2 \) and thus we must ensure that \( z_2 \) cannot perform a force on another vertex at the same time step. In order to prevent this, there must be a (largest) \( j > 1 \) such that \( q_j \in \text{out}(z_2) \). We let \( z_3 = q_j \). Now forcing may occur along \( Q \) until \( z_3 \) is colored blue.

We continue in this manner. At each stage \( z_k \) may not perform a force until \( z_{k+1} \) is blue. Once that occurs, we must ensure that \( z_{k+1} \) cannot perform a force at the same time step as \( z_k \). If neither \( z_{k-1} \) nor \( z_k \) are the last vertex of one of \( P \) or \( Q \), then \( \text{next}(z_{k-1}) \) and \( \text{next}(z_k) \) are forced blue at the same time step unless \( z_{k+1} \) is defined. Therefore, the process of creating the zig-zag path continues until we reach the end of one of \( P \) or \( Q \) for the first time. Call this vertex \( z_s \). By construction, we will have \( z_s = z_r \) or \( z_s = z_{r-1} \).

Suppose there is a vertex \( u \) for which \( u \in \text{out}(z_s) \) and \( z_i < u \) for all \( z_i \) in the same path as \( u \). Without loss of generality, we may assume that \( u \) is the last vertex in its path with this property. Then let \( z_s = z_{r-1} \) and \( u = z_r \). If \( z_r \neq \text{next}(z_{r-2}) \), then \( z_{r-1} \) will be able to force \( z_r \) blue at the same time step as \( z_{r-2} \) forces \( \text{next}(z_{r-2}) \). So, either \( z_s = \text{next}(z_{r-2}) \) or \( \text{prev}(z_r) \in \text{out}(z_{r-1}) \) so that \( z_{r-1} \) must wait to force \( z_r \) blue until \( \text{prev}(z_r) \) is blue. Therefore, we are in the situation described by condition 3(a). If there is no such \( u \), then let \( z_s = z_r \) and we are in the situation described by condition 3(b).

Conversely, assume \( B = \{p_1, q_1\} \) is a minimum zero forcing set and \( \overrightarrow{G} \) can be written as a zig-zag orientation \( \overrightarrow{G}(P, Q) \) with properties (1)-(3). Properties (1)-(3) require that only one force may occur at each time step and that \( P, Q \) are the forcing chains. Thus, \( \text{pt}(\overrightarrow{G}, B) = |G| - 2 \). □
The orientation $\overrightarrow{G}$ shown in Figure 3 gives an example for which $\text{pt}(\overrightarrow{G}) = |G| - 2$.

![Figure 3: An orientation satisfying Properties 1, 2, and 3b in Theorem 3.7.](image)

In order for $\text{pt}(\overrightarrow{G}) = |G| - 2$, Theorem 3.7 must hold true for all minimum zero forcing sets $B$ of size two.

**Corollary 3.8.** Let $\overrightarrow{G}$ be an orientation of a connected graph $G$ for which $Z(\overrightarrow{G}) = 2$. Then $\text{pt}(\overrightarrow{G}) = |G| - 2$ if and only if for every minimum zero forcing set $B$, $\overrightarrow{G}$ can be written as a zig-zag orientation $\overrightarrow{G}(P, Q)$ satisfying the three properties listed in Theorem 3.7.

In order to guarantee $\text{pt}(\overrightarrow{G}) = |G| - 2$, it is not simply enough to be able to write $\overrightarrow{G}$ as a zig-zag orientation $\overrightarrow{G}(P, Q)$ for some $P, Q$. Although it may be the case that $\text{pt}(\overrightarrow{G}, B) = |G| - 2$ for $B = \{p_1, q_1\}$, there may be another minimum zero forcing set $B'$ for which $\text{pt}(\overrightarrow{G}, B') < |G| - 2$. The following describe a few examples of zig-zag orientations $\overrightarrow{G}(P, Q)$ of $G$ for which $\text{pt}(\overrightarrow{G}) < |G| - 2$. That is, $B = \{p_1, q_1\}$ is a minimum zero forcing set for which $\text{pt}(\overrightarrow{G}, B) = |G| - 2$ yet there is another minimum zero forcing set $B'$ for which $\text{pt}(\overrightarrow{G}, B') < |G| - 2$. In these cases, we assume there are no back-arcs on either $P$ or $Q$ unless specified. However, some of the examples below are still valid with the addition of some (appropriate) back-arcs on $P$ and/or $Q$.

1. $\deg^+(p_1) = 1$, $z_r = q_t$, $(z_r, q_1)$ is an arc, and if $p_a \in \text{out}(q_t)$, then $p_a \prec z_2$. In this case, $B' = \{p_1, q_t\}$ is a zero forcing set for which $\text{pt}(\overrightarrow{G}, B') < |G| - 2$.

2. $\deg^+(p_1) = 1$, $z_r = q_t$, and arcs $(z_r, q_j), (q_j, q_1)$ exist where $z_{r-2} \prec q_j \prec z_r$. Then $B' = \{p_1, q_{j+1}\}$ is a zero forcing set for which $\text{pt}(\overrightarrow{G}, B') < |G| - 2$ provided there are no vertices $u \in \text{out}(q_b)$, $b = j + 2, \ldots, n$ such that $z_2 \prec u$.

3. $\deg^+(p_1) = 1$ and for $i > 2, 1 \leq j < i - 1$ there is a $p_i \prec z_2$, such that $(p_i, p_j)$ and $(p_{i-1}, q_1)$ are arcs and $\text{out}(p_i) = \emptyset$. In this case, $B' = \{p_1, p_i\}$ is a zero forcing set for which $\text{pt}(\overrightarrow{G}, B') < |G| - 2$ provided there are no vertices $p_a, a = 2, \ldots, i - 2$ such that $(p_a, q_1)$ is an arc.
4. \( \deg^+(p_1) = 1 \), \( q_1 \in \text{out}(p_s) \), and if there is an arc \((p_s, p_i)\), then \( p_i \prec z_2 \). In this case, \( B' = \{ p_1, p_s \} \) is a zero forcing set for which \( \text{pt}(G, B') < |G| - 2 \).

5. \( z_{r-1} = q_t \), \( z_r = p_s \), and \((z_r, p_i), (p_i, p_1), (z_{r-1}, q_j), (q_j, q_1)\) are arcs where \( p_i, q_j \) are such that one of the following hold:

   (a) \( z_{2k-2} \prec p_i \prec z_{2k+2} \) and \( z_{2k-1} \prec q_j \prec z_{2k+1} \) for \( k = 1, \ldots, r-2 \) (in this case, we call \( z_0 \) the vertex \( p_1 \)) and there are no vertices \( u \in \text{out}(p_a), a = i + 1, \ldots, s \) such that \( u \prec q_{j+1} \) or \( v \in \text{out}(q_b), b = j + 1, \ldots, t \) such that \( v \prec p_{i+1} \). The same is true of \( p_s \) and \( q_t \) with the exception that one of the arcs \((p_s, q_1)\) or \((q_t, p_1)\) may be present, but not both.

   (b) \( z_{r-2} \prec p_i \prec z_r, z_1 \prec q_j \prec z_3 \) and for \( a = i + 1, \ldots, m \) any vertices \( u \in \text{out}(p_a) \) are also in \((q_{j+1}, \ldots, z_3)\).

Then \( B' = \{ p_{i+1}, q_{j+1} \} \) is a zero forcing set for which \( \text{pt}(G, B') < |G| - 2 \).

Figures 4, 5, and 6 give examples of orientations of each of the five types listed above.

![Figure 4](image1.png)

Figure 4: Examples of orientations of types 1 and 2, respectively.

![Figure 5](image2.png)

Figure 5: Examples of orientations of types 3 and 4, respectively.

### 4 Low Propagation Times

The smallest possible propagation time would be 0. It is easy to see that in a connected oriented graph \( \overrightarrow{G} \) of order at least two, \( Z(\overrightarrow{G}) \leq |\overrightarrow{G}| - 1 \) and \( \text{pt}(\overrightarrow{G}) \geq 1 \). Thus the only oriented graph having propagation time equal to 0 is an oriented graph with no edges, i.e., a set of isolated vertices.
We now consider graphs that have an orientation that allows propagation time one. Between the zero forcing set and one iteration we must cover all vertices so we have the following observation.

**Observation 4.1.** For any oriented graph $\overrightarrow{G}$, if $pt(\overrightarrow{G}) = 1$, then $Z(\overrightarrow{G}) \geq \lceil |\overrightarrow{G}|/2 \rceil$.

Therefore the orientation must have a large zero forcing number (though this is not sufficient). Graphs with an orientation with propagation time one are not so easy to classify. Many graphs including trees (see Theorem 4.3) and complete graphs of order at least six (see Theorem 4.5) have such orientations. However, this is not true for all graphs, as seen in the following example, which shows there is no orientation of $K_4$ with propagation time one.

**Example 4.2.** All possible orientations of $K_4$, up to relabeling, are shown in Figure 7, taken from and labeled as in [11]. It is known that $Z(\overrightarrow{K_4}) = 1$ if $\overrightarrow{K_4}$ is a Hessenberg path (D149, with path (1, 2, 3, 4)), in which case $pt(\overrightarrow{K_4}) = 3$; for other orientations $Z(\overrightarrow{K_4}) = 2$ [4]. For D115 the only zero forcing set of cardinality two is $B_1 = \{1, 3\}$ and $pt(D115, B_1) = 2$. For D129 there are three possible zero forcing sets of cardinality two but they are all equivalent by symmetry to $B_2 = \{2, 3\}$, and $pt(D129, B_2) = 2$. Since D122 is the reverse of D129, $pt(D122) = 2$.

![Figure 7: Orientations on $K_4$ (D149, D115, D129 and D122 resp.)](image)

**4.1 Trees**

In this section we show that any tree (and hence any forest) can be oriented to have propagation time at most one, unless it is an isolated vertex.
Theorem 4.3. Let $T$ be a tree on $n \geq 2$ vertices. Then there is an orientation $\overrightarrow{T}$ of $T$ such that $\text{pt}(\overrightarrow{T}) = 1$.

Proof. A connected oriented graph $\overrightarrow{G}$ of order at least two has $\text{pt}(\overrightarrow{G}) \geq 1$, so it is sufficient to show that any tree $T$ of order at least two has an orientation $\overrightarrow{T}$ with $\text{pt}(\overrightarrow{T}) \leq 1$. We prove this statement by induction.

For the base case, it can be seen that $\text{pt}(\overrightarrow{T}) = 1$ when $n = 2$. We assume every nontrivial tree with fewer than $n$ vertices can be oriented to have propagation time one and consider a tree $T$ on $n$ vertices. Choose a vertex $y$ with $\deg(y) \geq 2$ such that at most one component of $T - y$ is a smaller tree $T'$ of order two or more; any other components are isolated vertices, which we denote by $z_1, z_2, \ldots, z_s$. If there is no component of order two or more, orient the edges of $T$ as $y \rightarrow z_i$, for $i = 1, 2, \ldots, s$, so $Z(\overrightarrow{T}) = n - 1$ and $\text{pt}(\overrightarrow{T}) = 1$. So assume there is a unique component $T'$ of order at least two. By the induction hypothesis, there is an orientation $\overrightarrow{T'}$ of $T'$ with $\text{pt}(\overrightarrow{T'}) = 1$. Let $B$ be an efficient minimum zero forcing set of $\overrightarrow{T'}$, and let $x$ denote the unique neighbor of $y$ among $V(T')$.

First suppose $x \notin B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T'}$ by orienting edges of $T$ not in $T'$ so that $\deg^+(y) = 0$, i.e., $x \rightarrow y$ and $y \leftarrow z_i$, $i = 1, 2, \ldots, s$. Observe that $B \cup \{z_i\}_{i=1}^s$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T})$.

Since $\deg^-(y) = 0$, $i = 1, \ldots, s$, a minimum zero forcing set of $\overrightarrow{T}$ must be of the form $B' \cup \{z_i\}_{i=1}^s$. In particular, $B'$ must force all vertices of $\overrightarrow{T'}$ without the help from $y$ or $\{z_i\}_{i=1}^s$, because $y$ cannot contribute any forces to $V(T')$. Therefore, $|B| \leq |B'|$ and

$$Z(\overrightarrow{T}) \leq |B \cup \{z_i\}_{i=1}^s| \leq |B' \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T}).$$

Next suppose $x \in B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T'}$ by orienting edges of $T$ not in $T'$ so that $\deg^-(y) = 0$, i.e., $x \leftarrow y$ and $y \rightarrow z_i$, for $i = 1, 2, \ldots, s$. Observe that $B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| = Z(\overrightarrow{T})$. Since $y$ can force at most one of $\{z_i\}_{i=1}^{s-1}$ to be blue, at least $s - 1$ of them must be blue initially. Without loss of generality, a minimum zero forcing set of $\overrightarrow{T}$ has the form $B' \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$. If $z_s \in B'$, then $(B' \setminus \{z_s\} \cup \{x\}) \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$ is a zero forcing set with the same cardinality. We further assume $x \in B'$ and then $B'$ forces all vertices of $\overrightarrow{T'}$ without the help of $\{y\} \cup \{z_i\}_{i=1}^{s-1}$. Therefore, $|B| \leq |B'|$ and

$$Z(\overrightarrow{T}) \leq |B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| \leq |B' \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| = Z(\overrightarrow{T}).$$

This completes the induction, and thus the proof. \hfill \Box

4.2 Tournaments

Trees are the sparsest connected graphs, i.e., those with the fewest possible number of edges. At the opposite extreme are tournaments, which are orientations of complete
graphs. However, we will see that for most $n$ there is a tournament on $n$ vertices that has propagation time one. In particular, we see that minimum propagation time is not strongly correlated with density.

**Proposition 4.4.** For $n \neq 2$, there is an orientation $\overrightarrow{K}_{2n}$ such that $\text{pt}(\overrightarrow{K}_{2n}) = 1$.

**Proof.** Since the tournament of order 2 has propagation time one, we will assume $n \geq 3$. Let $\mathbb{Z}_n$ be the additive cyclic group of order $n$. We split the vertices into two parts $U$ and $L$, and index the vertices by $\mathbb{Z}_n$, namely, $U := \{u_i : i \in \mathbb{Z}_n\}$ and $L := \{\ell_i : i \in \mathbb{Z}_n\}$. Place arcs between these vertices as follows: $A_1 = \{u_i \to \ell_{i-1} : i \in \mathbb{Z}_n\}$, and $A_2 = \{u_i \leftarrow \ell_j : j \neq i - 1\}$. Also, define $A_3 = \{u_i \to u_j : j - i \in H\}$, where $H = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \subseteq \mathbb{Z}_n$. $A_3$ is well-defined when $n$ is odd. When $n = 2k$ is even, each pair of $(u_i, u_{i+k})$ are doubly directed in $A_3$. In this case we modify $A_3$ by randomly choosing one of the directions $u_i \to u_{i+k}$ or $u_{i+k} \to u_i$ for each pair $(u_i, u_{i+k})$. Finally we define $A_4 = \{\ell_i \to \ell_j : u_i \to u_j\}$ and

$$E = A_1 \cup A_2 \cup A_3 \cup A_4.$$ 

Thus $\overrightarrow{K}_{2n} = (V, E)$ is a tournament on $2n$ vertices, where $V = U \cup L$. For convenience, we say $U$ is the upper part of $\overrightarrow{K}_{2n}$ and $L$ is the lower part of $\overrightarrow{K}_{2n}$.

We observe that $U$ forms a zero forcing set of $\overrightarrow{K}_{2n}$, $|U| = n$, and $\text{pt}(\overrightarrow{K}_{2n}, U) = 1$, so showing that $Z(\overrightarrow{K}_{2n}) = n$ completes the proof. In order to prove this, we let $S \subset V$ be a set of cardinality $n - 1$ and show it cannot be a zero forcing set.

**Case 1:** $S \subset L$.

By our assumption $n \geq 3$, every blue vertex has at least one white out-neighbor in $U$, so $S$ cannot be a zero forcing set.

**Case 2:** $S \subset U$.

Since $|S| = n - 1$, by symmetry we assume $u_0$ is the only vertex in $U \setminus S$. Let $X := S \cap N^-(u_0)$ and $Y := S \setminus X$. At the first time step, the vertices in $Y$ can force downward to color the set $Y_L \subset L$ blue; observe $\ell_{n-1} \notin Y_L$ because the only in-neighbor of $\ell_{n-1}$ in $U$ is $u_0 \notin S$. At this time, no $\ell_i \in Y_L$ can force any $\ell_j$ since $u_0$ is a white out-neighbor of $\ell_i$ (since $i \neq n - 1$). Also, we observe that every vertex in $Y$ has no white out-neighbors whereas every vertex in $X$ has two white out-neighbors: One is $u_0$ and one is in $L$. Therefore the only possibility for an additional force is $\ell_i$ forces $u_0$ for some $i$.

Now we consider two cases and show each is impossible. First if $n = 2k + 1$ is odd, then $\ell_i$ has $k$ out-neighbors in $L$. But $Y_L$, the set of blue vertices in $L$, contains only $k$ vertices, including $\ell_i$ itself. So $\ell_i$ must have another white out-neighbor in $L$, making it impossible to force $u_0$. Second, if $n = 2k$ is even, then $|Y|$ can be either $k - 1$ or $k$. If it is $k - 1$, then we apply the same argument for odd numbers. So without loss of generality, we assume $u_k$ is an out-neighbor of $u_0$ and thus $Y = \{u_1, u_2, \ldots, u_k\}$. By our construction, $Y_L$ will be $\{\ell_0, \ell_1, \ldots, \ell_{k-1}\}$ and $\ell_k$ is an out-neighbor of $\ell_0$. Under this assumption, for all $i \neq n - 1$, $\ell_i$ has at least one white out-neighbor in $L$, and
$u_0 \in N^+(\ell_i)$. Thus for $i \neq n - 1$, $\ell_i$ cannot force. All of the out-neighbors of $\ell_{n-1}$, are blue, so $\ell_{n-1}$ cannot force. Thus $S$ is not a zero forcing set.

**Case 3:** $S \cap U$ and $S \cap L$ are not empty.

We start by carrying out the following shift process: If some $u_i \in S \cap U$ has only one white out-neighbor and it is in $U$, called $u_j$, then we replace $S$ by $S - \ell_{i-1} + u_j$.

Since in the new set $u_i$ can force $\ell_{i-1}$, it is sufficient to say this new set is not a zero forcing set. Continuing this process, we may assume the set $S$ has the property that if $\ell_{i-1}$ is blue then either $u_i$ is white or it has two white out-neighbors in $U$. After completing the shift process, assume $S \cap U$ and $S \cap L$ are not empty, else we may apply Case 1 or Case 2.

Under our assumption, we claim that $|S \cap U| = n - 2$ and $|S \cap L| = 1$. First observe that $|S \cap U| \neq n - 1$, for otherwise $S \cap L = \emptyset$. Suppose $|S \cap U| \leq n - 3$.

Then there are at least three white vertices are white at the beginning in $U$. Every vertex in $L$ will have at least two white out-neighbor in $U$; every vertex cannot force these three vertices, since we completed the shift process. Therefore, $S$ cannot be a zero forcing set, since $S \cup L$ is not.

Now we assume $U - S = \{u_0, u_j\}$ for some $j$, and $S \cap L = \{\ell_i\}$ for some $i$. Denote $N_0$ as the out-neighbors of $u_0$ in $U$, and $N_j$ as the out-neighbors of $u_j$ in $U$. Let $Y = N_0 \cap N_j$ and $X = S - Y$. Note that $u_{i+1} \notin Y$, for otherwise $S - \ell_i$ is also a zero forcing set, and it is contained in $U$. Therefore, $Y$ can force a set $Y_L \subset L$ to be blue, and $|Y| = |Y_L|$. If $n = 2k + 1$ is odd, then $|N_0| = |N_j| = k$ but they are not the same, so $|Y| = |Y_L| \leq k - 1$. This means every blue vertex in $L$ have a white out-neighbor in $L$ and at least $\ell_i$ has at least one white out-neighbor in $U$. The process stops and $S$ is not a zero forcing set. On the other hand, if $n = 2k$, then it is either $|N_0| = |N_{i+1}| = k$ but they are not the same, or at least one of them is $k - 1$. In either cases, we have $|Y| = |Y_L| \leq k - 1$. If $|Y_L| < k - 1$, then similarly every blue vertex in $L$ have a white out-neighbor in $L$ and at least one white out-neighbor in $U$. So we assume $|Y_L| = k - 1$, and this happens only when $j = n - 1$ or $j = 1$. But this two cases are equivalent by symmetry, so we assume $j = 1$. In this case, $Y = \{u_2, \ldots, u_k\}$ and $Y_L = \{\ell_1, \ldots, \ell_{k-1}\}$. Since we assume $n \geq 3$, $k \geq 2$ and $u_0 \rightarrow u_k$ must be an arc. That means $\ell_0 \rightarrow \ell_k$ is also an arc; no matter what the initial blue vertex $\ell_i$ in $L$ it is, the process stops.

Throughout all cases, $S$ cannot be a zero forcing set.

**Theorem 4.5.** For all integers $n \geq 2$, $n \neq 4, 5$, there is an orientation $\overrightarrow{K}_n$ for $K_n$ such that $\text{pt}(\overrightarrow{K}_n) = 1$.

**Proof.** We have already seen that this statement is true for even $n$. For odd $n = 2m + 1$, we construct $\overrightarrow{K}_{2m+1}$ by adding one vertex $x$ to an orientation of $\overrightarrow{K}_{2m}$ constructed in Proposition 4.4, adding arcs from $x$ to all vertices in $\overrightarrow{K}_{2m}$.

Since the case $\overrightarrow{K}_3$ is trivial, we assume $m \geq 3$. In this case, every vertex in $\overrightarrow{K}_{2m}$ has in-degree at least one under the construction in Proposition 4.4. Let $B$
be a minimum zero forcing set for $\vec{K}_{2m+1}$. Since $\deg^- (x) = 0$, $x \in B$. Since $\deg^+ (x) = 2m$, $x$ cannot perform a force until all but one vertex of $\vec{K}_{2m} = \vec{K}_{2m+1} - x$ are blue, at which point another vertex can perform the force, since $\deg^- (v) \geq 1$ for every vertex $v \in \vec{K}_{2m}$. So $B \setminus \{x\}$ is a zero forcing set for $\vec{K}_{2m}$, implying $|B \setminus \{x\}| \geq Z(\vec{K}_{2m}) = m$ and $|B| \geq m + 1$. Therefore the propagation time is one.

4.3 Data for small graphs which allow propagation time one

Let us say that a simple graph $G$ allows propagation time one if there is some orientation of the graph, $\vec{G}$, with $pt(\vec{G}) = 1$. Then a natural question is which graphs allow propagation time one, and in general, how common are such graphs.

For all connected graphs of order up through nine, the minimum propagation time over all orientations was determined, and we present the data in Table 1 (no graph had a minimum propagation time of three or greater). At least for small graphs it appears that allowing propagation time one is common.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\min_{\vec{G}} pt(\vec{G})=1 & 1 & 2 & 5 & 20 & 106 & 820 & 10746 & 256568 \\
\min_{\vec{G}} pt(\vec{G})=2 & 0 & 0 & 1 & 1 & 6 & 33 & 371 & 4512 \\
\hline
\end{array}
\]

Table 1: Number of connected graphs on $n$ vertices with given minimum propagation time

We have already established that $K_4$ and $K_5$ are graphs that do not allow propagation time 1. In Figure 8 we give orientations for all the remaining graphs on at most 4 vertices, and have marked corresponding minimum zero forcing sets, to verify that they have $pt(\vec{G}) = 1$. We remark here that the undirected graph underlying the oriented graph in Figure 1 does not allow an orientation with propagation time one.

![Figure 8: The connected graphs smaller than $K_4$ with orientations having propagation time one](image-url)
In addition to the data given in the table, we have verified by computer that no graph of order 10 requires propagation time of three or greater. This leads to the following open question.

**Question 4.6.** Does there exist an undirected graph $G$ so that $\min_{\rightarrow G} \text{pt}(G) \geq 3$, where the minimum runs over all orientations of $G$? More generally, for each $k$ does there exist an undirected graph $G$ so that $\min_{\rightarrow G} \text{pt}(G) \geq k$, where the minimum runs over all orientations of $G$?

## 5 Orientation Propagation Intervals

We have seen in preceding sections that for the path $P_n$ there are orientations with both high and low propagation times. This leads to the following idea.

**Definition 5.1.** Let $G$ be an undirected graph with $m = \min_{\rightarrow G} \text{pt}(G)$ and $M = \max_{\rightarrow G} \text{pt}(G)$. The interval $[m, M]$ is called the orientation propagation interval, and $G$ has a full orientation propagation interval if for every $m \leq k \leq M$ there is some orientation $\rightarrow G$ such that $\text{pt}(G) = k$.

Determining if a graph has a full orientation propagation interval is non-trivial, even for some simple graphs. The difficulty is that the propagation parameter can be sensitive to small perturbations, as shown in the following example.

**Example 5.2.** Let $n \geq 9$ and $k = \left\lceil \frac{n+3}{2} \right\rceil$. Consider the two oriented paths on $n$ vertices shown in Figure 9, where the vertices are labeled 1 to $n$ going from left to right. The top path has $Z(\rightarrow P_n) = 2$ and $\{1, k\}$ is the minimum zero forcing set with fastest propagation time $n - 2$ (in fact $\{1, k\}$ is the unique minimum zero forcing set). The bottom path has $Z(\rightarrow P_n) = 3$ and $\{1, k, k + 1\}$ is the minimum zero forcing set with fastest propagation time of $\left\lceil \frac{n-5}{2} \right\rceil$. Thus the reversal of the arc between $k - 2$ and $k - 1$ changed the propagation time by $\left\lfloor \frac{n+1}{2} \right\rfloor$, which can become arbitrarily large.

![Figure 9: Reversing an arc produces a large change in propagation time](image)

In this section we will show that paths have full orientation propagation time interval, while cycles do not. We comment that the behavior and analysis of orientation propagation time intervals is far from understood.
5.1 Paths have a full orientation propagation time interval

By Theorem 4.3 we know for $P_n$ that there is an orientation with propagation time one, and if we orient the edges of the path as $(i, i + 1)$ for $1 \leq i \leq n - 1$, then the propagation time is $n - 1$. We will show that for $P_n$ there is an orientation with propagation time $k$ for each $1 \leq k \leq n - 1$.

To achieve a propagation time of $n - 2$, take the orientation $(i, i + 1)$ for $1 \leq i \leq n - 3$ together with $(n - 1, n - 2)$ and $(n - 1, n)$. Then this has zero forcing number 2 with unique minimal zero forcing set $\{1, n - 1\}$, further no simultaneous forces can occur giving us propagation time $n - 2$.

The remaining propagation times are given by the following theorem.

**Theorem 5.3.** Let $P_n$ be the path on $n$ vertices and $2 \leq k \leq n - 3$. Then there is an orientation $\overrightarrow{P}_n$ such that $pt(\overrightarrow{P}_n) = k$.

**Proof.** We label the path with vertices 1, $\ldots$, $n$ and edges joining $i$ and $i + 1$ for $1 \leq i \leq n - 1$. We consider the following orientation.

- $(i, i + 1)$ for $1 \leq i \leq k + 1$ (the initial segment).
- For $k + 2 \leq j \leq n - 1$ orient the edge between $j$ and $j + 1$ by

$$
\begin{cases}
(j, j + 1) & \text{if } j \equiv k \text{ or } k + 1 \pmod{4}, \\
(j + 1, j) & \text{if } j \equiv k + 2 \text{ or } k + 3 \pmod{4}.
\end{cases}
$$

A minimum zero forcing set must contain 1, but no other vertex in the initial segment (as 1 can eventually force the initial segment). In particular, vertex $k + 1$ will not turn blue until the $k$th step of the propagation (the vertex $k + 2$ can be turned blue earlier through its other neighbor). Therefore the propagation time of this orientation is at least $k$.

Consider the set $S = \{i : \deg^-(i) = 0\}$. The vertices in $S$ must be in a zero forcing set since they cannot be turned blue by a neighbor. If a vertex in $S$ has $\deg^+(i) = 2$ then one of the neighbors must also be in the zero forcing set, i.e., only $i$ can change them to blue, but it cannot force both. As a consequence when we look at blocks of consecutive vertices between vertices with $\deg^-(i) = 2$ we see that each block will have two elements in the zero forcing set and the block will propagate in time two. Similar analysis shows that the tail will also propagate in time at most two.

We conclude that in two steps all but the initial segment of the path has been turned blue, and that will not finish turning blue until time $k$ and so the propagation time of this orientation of $P_n$ is $k$. \qed
5.2 Cycles do not have a full orientation propagation time interval

Not all graphs have a full propagation orientation interval, as the following example shows.

Example 5.4. The four orientations on $C_4$ up to isomorphism are shown in Figure 10. No orientation has a propagation time of 2, although there are orientations with propagation times 1 and 3. So $C_4$ does not have a full propagation orientation interval.

If we orient the cycle $C_n$ by $(i, i + 1)$ for $1 \leq i \leq n$ (where we look at the entries modulo $n$), then any vertex can force the entire graph and has $pt(C_n) = n - 1$. If we now reverse the arc between 1 and $n$ to $(1, n)$, then by using the zero forcing set $\{1, 2\}$ in the first step we force 3 and $n$ and then at each subsequent step we force one vertex and so this has propagation time $n - 3$, further any other zero forcing set for this orientation has propagation time $n - 2$. So $pt(C_n) = n - 3$. However the intermittent value of $n - 2$ is impossible as the following result shows. In particular, the cycle $C_n$ for $n \geq 4$ does not have a full orientation propagation time interval.

Proposition 5.5. Let $n \geq 4$. Then $pt(C_n) \neq n - 2$ for any orientation of $C_n$.

Proof. Suppose $C_n$ is an orientation of $C_n$ with $pt(C_n) = n - 2$. By Observation 3.2 we must have that $Z(C_n) = 2$. Moreover it must be the case that precisely one vertex is forced at each time step for every minimum zero forcing set. Since $Z(G) = 2$, $G$ is a graph of two parallel Hessenberg paths, with two arcs between the two paths so that a cycle is formed. Since the two end vertices $v$ and $w$ must be a zero forcing set and cannot both force initially, without loss of generality the arcs must be oriented as shown in Figure 11.

Then if an out-neighbor $u$ of $v$ exists in the path containing $v$, then $\{v, u\}$ is a zero forcing set in which two forces are performed initially, contradicting $pt(G) = n - 2$. If $u$ does not exist, then $\{v, w\}$ is a zero forcing set in which two forces are performed initially (since $n \geq 4$, $y$ is not the last vertex in the lower path), contradicting $pt(G) = n - 2$. \qed
6 Throttling on Oriented Graphs

To this point we have focused on the propagation time for zero forcing sets that have minimum cardinality. We can relax and more generally look at any sets that force the entire graph. In this situation we want to balance both the size of the set and the speed at which it propagates through the graph.

**Definition 6.1.** Given an oriented graph \( \vec{G} \) and a zero forcing set \( B \) of \( \vec{G} \), the **throttling time** of \( B \) for \( \vec{G} \) is \( \text{th}(\vec{G}, B) := \text{pt}(\vec{G}, B) + |B| \). The **minimum throttling time** of an oriented graph \( \vec{G} \) is \( \text{th}(\vec{G}) = \min \{ \text{pt}(\vec{G}, B) + |B| : B \text{ is a zero forcing set of } \vec{G} \} \).

For undirected graphs throttling was studied in [6] where the throttling time of a path was determined and more generally it was shown that for any fixed value \( k \) that if the minimum zero forcing set for a graph on \( n \) vertices had size at most \( k \), then the minimum throttling time was of order \( c\sqrt{n} \).

We look at throttling on complete Hessenberg paths. A **complete Hessenberg path** is the unique tournament with a zero forcing number of one. More precisely, for the graph with vertex set \( \{1, 2, \ldots, n\} \) the complete Hessenberg path has the following arcs:

\[
\{(i, i+1), (i, j) : 1 \leq i \leq n - 1 \text{ and } 1 \leq j < i - 1\}.
\]

A simple check verifies that for \( n \geq 4 \) that \( \{1\} \) is the unique minimum zero forcing set of this oriented graph. We show an example in Figure 12 when \( n = 5 \).

![Figure 12: A complete Hessenberg path on 5 vertices.](image)

We will show that unlike undirected graphs, we cannot guarantee a significant savings in throttling. In particular, for the complete Hessenberg path \( \vec{H} \), we have \( \text{th}(\vec{H}) = \lfloor 2n/3 \rfloor + 1 \). The key step is given in the next lemma which shows that we cannot engage in a large number of simultaneous forces on the complete Hessenberg.

**Lemma 6.2.** Let \( \vec{H} \) be a complete Hessenberg path, and \( B \) a set of blue vertices. Then \( B \) can force at most 2 vertices at any given timestep.
Proof. Let $\overrightarrow{H}$ be a complete Hessenberg path with vertex set $\{1, 2, \ldots, n\}$ and assume $B$ is a set of blue vertices that forces 3 or more vertices at timestep $t$. Assume $a < b < c$ are the largest of the vertices that are forced at time $t$ and thus are white at time $t - 1$. Observe that $c$ can only be forced by vertex $c - 1$ or by some vertex $c + 2$ or greater; but any vertex $c + 2$ or greater has $a, b$ and $c$ as white out-neighbors so cannot perform any forces. This means $c - 1$ must force $c$. However, $a < b \leq c - 1$ so $c - 1$ has both $a$ and $c$ as white out-neighbors so cannot perform any forces. This means that no vertex can force vertex $c$ which is a contradiction and the result follows. \hfill $\square$

**Corollary 6.3.** For any zero forcing set $B$ of the complete Hessenberg path $\overrightarrow{H}$, we have $2 \text{pt}(\overrightarrow{H}, B) + |B| \geq n$.

In the remainder of this section we will find it convenient for the proofs to group the vertices of the complete Hessenberg path on $n$ vertices into sets of three. We will adopt the following notation $\ell := \lfloor n/3 \rfloor$ and for $1 \leq j \leq \ell$ then $I_j = \{3j - 2, 3j - 1, 3j\}$ while $I_{\ell+1}$ will be the remaining vertices (if any). We also note that any zero forcing set must contain at least one vertex in $I_1$, i.e., if not then since every vertex is adjacent to two elements in $I_1$ then we could never change $I_1$ to blue.

**Proposition 6.4.** If $\overrightarrow{H}$ is a complete Hessenberg path with $|\overrightarrow{H}| = n$ then $\text{th}(\overrightarrow{H}) \leq \lfloor 2n/3 \rfloor + 1$.

*Proof. Define $B := \{1, 3, 6, 9, \ldots, 3\ell\}$ and note that $|B| = \lfloor n/3 \rfloor + 1$. An easy inductive argument show that $I_j$ will be blue at time $j$ for $1 \leq j \leq \ell$ and $I_{\ell+1}$ turns blue at time $\ell$ or $\ell + 1$ if $|I_{\ell+1}| \leq 1$, or $|I_{\ell+1}| = 2$ respectively. If $n \equiv 0 \mod 3$ or $n \equiv 1 \mod 3$ then $\text{pt}(\overrightarrow{H}, B) = \lfloor n/3 \rfloor$ and $\text{th}(\overrightarrow{H}, B) = 2 \lfloor n/3 \rfloor + 1 = \lfloor 2n/3 \rfloor + 1$. If $n \equiv 2 \mod 3$ then $\text{pt}(\overrightarrow{H}, B) = \lfloor n/3 \rfloor + 1$ and $\text{th}(\overrightarrow{H}, B) = 2 \lfloor n/3 \rfloor + 2 < \lfloor 2n/3 \rfloor + 2$. In all cases $\text{th}(\overrightarrow{H}, B) \leq \lfloor 2n/3 \rfloor + 1$. \hfill $\square$

**Lemma 6.5.** If $\overrightarrow{H}$ is a complete Hessenberg path on $n$ vertices and $B$ is a zero forcing set such that $|B| \leq \lfloor n/3 \rfloor$ then there is some time step when exactly one vertex is forced.

*Proof. Suppose first that for some $2 \leq m \leq \ell$ that $I_m \cap B = \emptyset$. We will let $S = I_1 \cup \cdots \cup I_{m-1}$ and $T = I_m \cup \cdots \cup I_{\ell+1}$. Since every blue vertex in $T$ is adjacent to two white vertices in $I_m$ then nothing in $T$ can force until $3m - 2$ (the first element of $T$) has been forced blue and further when $3m - 2$ has been forced everything in $S$ has also already been forced (in order for $3m - 3$ to force then $1, 2, \ldots, 3m - 5$ must be blue; if $3m - 4$ is blue before $3m - 3$ forces then the result holds and otherwise $3m - 5$ will force $3m - 4$ at the same time that $3m - 3$ forces $3m - 2$). At this stage $3m - 2$ can force $3m - 1$ and the only other vertex which can possibly be adjacent to only one white vertex is $3m + 1$ adjacent to $3m - 1$. Therefore the only vertex which is forced at this stage is $3m - 1$. \hfill $\square$
If all of the $I_m \cap B \neq \emptyset$ then it must be that $|I_m \cap B| = 1$ for $1 \leq m \leq \ell$. Therefore every vertex which is 4 or greater has at least two white out-neighbors and so only the single blue vertex in $I_1$ can force.

**Theorem 6.6.** If $\vec{H}$ is a complete Hessenberg path then $\text{th}(\vec{H}) = \lceil 2n/3 \rceil + 1$.

**Proof.** An easy verification establishes the result when the complete Hessenberg path $\vec{H}$ has 1, 2 or 3 vertices. We will now proceed by induction on $n$, the number of vertices of $\vec{H}$. Assume that $\text{th}(\vec{H}) = \lceil 2k/3 \rceil + 1$ for $1 \leq k \leq n - 1$. Consider the complete Hessenberg path on $n$ vertices and for the sake of contradiction we will assume we can find a zero forcing set $B$ of $\vec{H}$ with $\text{th}(\vec{H}, B) \leq \lceil 2n/3 \rceil$. Then by Corollary 6.3 and this assumption we have

$$n \leq \text{pt}(\vec{H}, B) + \text{pt}(\vec{H}, B) + |B| \leq \text{pt}(\vec{H}, B) + \lceil 2n/3 \rceil \leq \text{pt}(\vec{H}, B) + 2n/3.$$  

This tells us that $n/3 \leq \text{pt}(\vec{H}, B)$ which in turn implies that $|B| \leq \lceil n/3 \rceil$. We now consider two cases.

**Case 1:** $I_m \cap B = \emptyset$ for some $2 \leq m \leq \ell$

Proceeding as in the previous lemma we let $S = I_1 \cup \cdots \cup I_{m-1}$ and $T = I_m \cup \cdots \cup I_{\ell+1}$ and note that no force happens from the blue vertices in $T$ until all vertices in $S$ and $x := 3m - 2$ (the first vertex in $T$) have been forced. Once $x$ has been forced the rest of the forcing can proceed as though $S$ is not part of the graph. This shows we can split the propagation into two distinct phases and so we have

$$\text{pt}(\vec{H}, B) = \text{pt}(\vec{H}[S \cup \{x\}], B \cap S) + \text{pt}(\vec{H}[T], (B \cap T) \cup \{x\}).$$

Using this we get the following bound:

$$\text{th}(\vec{H}, B) = \text{pt}(H, B) + |B|$$

$$= \text{pt}(\vec{H}[S \cup \{x\}], B \cap S) + \text{pt}(\vec{H}[T], (B \cap T) \cup \{x\}) + |B \cap S| + |B \cap T|$$

$$= \text{th}(\vec{H}[S \cup \{x\}], B \cap S) + \text{th}(\vec{H}[T], (B \cap T) \cup \{x\}) - 1$$

$$\geq [2(|S| + 1)/3] + 1 + [2(n - |S|)/3] + 1 - 1$$

$$\geq [2n/3] + 1$$

The $-1$ term on the third line comes from accounting for $\{x\}$, and on the fourth line we have used the induction hypothesis. This statement contradicts that $\text{th}(\vec{H}, B) \leq \lceil 2n/3 \rceil$, so this case cannot happen.

**Case 2:** $I_m \cap B \neq \emptyset$ for all $1 \leq m \leq \ell$

In this case we must then have that $|B| = \lceil n/3 \rceil$ and so

$$\text{th}(\vec{H}, B) \geq 1 + \frac{n - \lceil n/3 \rceil - 1}{2} + \lceil n/3 \rceil,$$
where the first two terms are a lower bound for $pt(\vec{H}, B)$ since by Lemma 6.5 there is some point in the forcing process when only one force occurred and since for every other point at most two forces could occur. Since $th(\vec{H}, B)$ must be a whole number this then implies that $th(\vec{H}, B) \geq \lceil 2n/3 \rceil + 1$ which is again a contradiction to $th(\vec{H}, B) \leq \lceil 2n/3 \rceil$.

Therefore we can conclude that $th(\vec{H}, B) \geq \lceil 2n/3 \rceil + 1$ and the construction in Proposition 6.4 is tight.

References


