On the tree cover number of a graph

Chassidy Bozeman∗ Minerva Catral† Brendan Cook‡ Oscar E. González§ Carolyn Reinhart¶∥

Abstract

Given a graph $G$, the tree cover number of the graph, denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs that cover all the vertices of $G$. This graph parameter was introduced in 2011 as a tool for studying the maximum positive semidefinite nullity of a graph, and little is known about it. It is conjectured that the tree cover number of a graph is at most the maximum positive semidefinite nullity of the graph.

In this paper, we establish bounds on the tree cover number of a graph, characterize when an edge is required to be in some tree of a minimum tree cover, and show that the tree cover number of the $d$-dimensional hypercube is 2 for all $d \geq 2$.

Keywords: graphs, matrices, tree cover number, maximum positive semidefinite nullity
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1 Introduction

A simple graph is a pair $G = (V,E)$, where $V = \{1, 2, \ldots, n\}$ is the vertex set, and $E$, the edge set, is a set of 2-element subsets (edges) of the vertices. A multigraph is a pair $G = (V,E)$, where $V = \{1, 2, \ldots, n\}$, and $E$ is a multiset of 2-element subsets of the vertices. That is, a multigraph allows multiple edges between a pair of vertices. (Note that all simple graphs are multigraphs.) Two
vertices $u, v \in V(G)$ are said to be adjacent if $\{u, v\} \in E(G)$. We say that the edge $\{u, v\} \in E(G)$ is a simple edge if $\{u, v\}$ appears in $E(G)$ exactly once. If $\{u, v\}$ appears in $E(G)$ more than once, then it is a multiedge. All graphs in this paper are considered to be multigraphs unless otherwise stated.

For a multigraph $G$, $S(G)$ denotes the set of real valued symmetric $n \times n$ matrices $(a_{i,j})$ satisfying:
1. $a_{i,j} = 0$ if $i \neq j$ and $i, j$ are nonadjacent,
2. $a_{i,j} \neq 0$ if $i \neq j$ and $i, j$ are adjacent via one edge, and
3. $a_{i,j} \in \mathbb{R}$ if $i = j$ or $i, j$ are adjacent via multiple edges.

The maximum nullity of a multigraph $G$ is defined to be
$$M(G) = \max \{\text{null}(A) : A \in S(G)\}.$$  

The maximum nullity of a simple graph $G$ is equivalent to the maximum multiplicity of an eigenvalue among all matrices in $S(G)$. This graph parameter has connections to many other concepts in linear algebra (as can be seen in [5] and [4]), and has been given a significant amount of consideration as it is very difficult to compute.

A related and equally important parameter is the maximum positive semidefinite nullity of a graph. A symmetric $n \times n$ real matrix $A$ is said to be positive semidefinite if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. The maximum positive semidefinite nullity of a multigraph $G$ is defined to be
$$M_+(G) = \max \{\text{null}(A) : A \in S_+(G)\},$$  

where $S_+(G) = \{A \in S(G) : A \text{ is positive semidefinite}\}$. It follows that for a multigraph $G$, $M_+(G) \leq M(G)$. In some cases, one can use tools such as orthogonal representations (see [5]) to compute $M_+(G)$, obtaining a lower bound for $M(G)$.

The tree cover number of a graph was introduced in 2011 in [1] as another tool for studying the maximum positive semidefinite nullity of a multigraph.

The (simple) path on $n$ vertices, denoted $P_n$, is the graph with vertex set $V(P_n) = \{1, \ldots, n\}$ and edge set $E(P_n) = \{\{i, i+1\} | i = 1, \ldots, n-1\}$. A simple graph $G = (V, E)$ is said to be a tree if for every $u, v \in V(G)$, there is exactly one path from $u$ to $v$.

Given a graph $G = (V, E)$, a subgraph, $G' = (V', E')$, is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, i.e, a subgraph of a graph $G$ can be obtained by deleting edges and/or vertices (and edges incident to the deleted vertices) of $G$. A subgraph $G' = (V', E')$ of $G$ is said to be an induced subgraph of $G$ if for each edge $uv \in E(G)$ with $u, v \in V(G')$, it follows that $uv \in E(G')$, i.e, an
induced subgraph of $G$ can be obtained by only deleting vertices (and any edges incident to the deleted vertices). For a subset $S \subseteq V(G)$, the graph induced by $S$, denoted $G[S]$, is the induced subgraph of $G$ with vertex set $S$.

A tree cover is a set of vertex disjoint simple trees occurring as induced subgraphs that cover all the vertices of the graph. The tree cover number of a graph $G$, denoted $T(G)$, is defined as

$$T(G) = \min\{|T| : T \text{ is a tree cover of } G\}.$$ 

Conjecture 1. $\overline{T(G)} \leq M_4(G)$.

This bound has been proven to be true for several families of graphs, including outerplanar graphs [1] and chordal graphs [1]. In fact, equality holds for outerplanar graphs [1] (and in fact for all graphs of tree-width at most 2, as observed in [2]).

In section 2 we give bounds on the tree cover number, provide an example in which the tree cover number behaves like the maximum positive semidefinite nullity, and provide an example in which the tree cover number does not behave like the maximum positive semidefinite nullity. See [1] and [3] for definitions of outerplanar and tree-width. In section 3 we characterize when an edge is required to be in some tree of a minimum tree cover. In section 4 we prove that the tree cover number of the $d$-dimensional hypercube is 2 for all $d \geq 2$.

1.1 More Notation and Terminology

The cycle on $n$ vertices, denoted $C_n$, is the graph with vertex set $V(C_n) = \{1, \ldots, n\}$ and edge set $E(C_n) = \{\{i, i+1\} | i \in 1, \ldots, n-1\} \cup \{1, n\}$. The star $K_{1,n}$ is the graph with vertex set $\{1, \ldots, n\}$ and edge set $\{\{1, j\} | j \in 2, \ldots, n\}$. The complete graph, denoted $K_n$, is the graph on $n$ vertices such that there is an edge between any two vertices.

A graph is said to be connected if there is a path from any vertex to any other vertex. If $G$ is not connected, then it is said to be disconnected. Given a graph $G = (V,E)$, a connected component of $G$ is a subgraph, $C$, where $C$ is connected and no vertex in $C$ is adjacent to any vertex of $V(G) \setminus V(C)$. A graph is said to be a forest if each of its connected components is a tree.

If vertices $u$ and $v$ are adjacent, we say that they are neighbors. The neighborhood of a vertex $v$, denoted $N(v)$, is the set of neighbors of $v$. The degree of $v$ is given by $\deg(v) = |N(v)|$.

For a graph $G = (V,E)$, a cover of $G$ is a partition of $V(G)$. An independent set, $S$, is a subset of $V(G)$ such that no two vertices in $S$ are adjacent. The
In connection with the conjecture that $T(G)$ is the independence number of $G$, denoted $\alpha(G)$, is defined by

$$\alpha(G) = \max\{|S| : S \text{ is an independent set in } G\}.$$

Given two simple graphs $G$ and $H$, the cartesian product of $G$ and $H$, denoted $G \times H$ is the graph whose vertex set is the cartesian product $V(G) \times V(H)$, and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$. The union of $G$ and $H$, denoted $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Throughout this paper, we often denote an edge $\{u, v\}$ by $uv$. An edge $uv$ is called a bridge of $G$ if $C - uv$ is disconnected, where $C$ is the component of $G$ with $uv \in E(C)$ and $C - uv$ denotes the subgraph obtained from $C$ by deleting the edge $uv$. Note that if $e = uv$ is a bridge, then $e = uv$ is a simple edge.

### 2 Some Bounds for the Tree Cover Number

In this section, we give an upper bound on the tree cover number of a graph using the size of an independent set in the graph. We also provide upper and lower bounds on the tree cover number of a subgraph of $G$ obtained by deleting an edge from $G$. In addition, we observe that subdividing an edge of a graph does not change the tree cover number.

The following proposition shows that, for a connected simple graph, we are able to bound the tree cover number by the difference between the order of the graph and the size of an independent set of vertices of the graph.

**Proposition 2.** Let $G = (V, E)$ be a connected simple graph, and let $S \subseteq V(G)$ be an independent set. Then, $T(G) \leq |G| - |S|$. In particular, $T(G) \leq |G| - \alpha(G)$, where $\alpha(G)$ is the independence number of $G$. Furthermore, this bound is tight.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and suppose that $S = \{v_1, \ldots, v_k\}$ is an independent set. We construct a tree cover of size $n - k$ by the following iterative process: For $i = k + 1$, let $T_{v_i}$ be the tree induced by the set of vertices $\{v_{k+1}\} \cup \{N(v_{k+1}) \cap S\}$. For $i = k + 2$ to $n$, let $T_{v_i}$ be the tree induced by the set of vertices in $\{v_i\} \cup \{N(v_i) \cap S\}$ that do not belong to $V(T_{v_{j}})$ for $k + 1 \leq j < i$. Since $G$ is connected, each $s \in S$ has at least one neighbor in $\{v_{k+1}, \ldots, v_n\}$, so this process produces a tree cover of $G$ (where all components are stars) of size $n - k$. Thus, $T(G) \leq n - k$. In particular, $T(G) \leq n - \alpha(G)$. The star $K_{1,n}$ shows that the bound $T(G) \leq |G| - \alpha(G)$ is tight.

In connection with the conjecture that $T(G) \leq M_+(G)$, we show that for some bounds on $M_+(G)$, analogous bounds hold for $T(G)$.
For a graph $G = (V, E)$ and $e \in E(G)$, let $G - e$ denote the graph obtained from $G$ by deleting the edge $e$. In [2], it was shown that

$$M_+(G) - 1 \leq M_+(G - e) \leq M_+(G) + 1,$$

when $G$ is a simple graph. We show that an analogous bound holds for the tree cover number of a multigraph $G$.

**Theorem 3.** For a graph $G = (V, E)$ and $e \in E(G)$,

$$T(G) - 1 \leq T(G - e) \leq T(G) + 1.$$

**Proof.** Let $u, v \in V(G)$ such that $e = uv$. Consider the graph $G - e$ obtained from $G$ by deleting $e$ (note that $e$ could be a multiedge). Let $T$ be a minimum tree cover of $G - e$. If $u$ and $v$ are in disjoint trees in $T$, then $T$ is a tree cover of $G$. So, $T(G) \leq T(G - e)$. If $u$ and $v$ are in the same tree in $T$, denoted by $T_{uv}$, then the graph induced by the vertices of $T_{uv}$ contains a cycle in $G$, so $T$ is not a tree cover of $G$. However, we may partition the vertices of $T_{uv}$ into two sets $A$ and $B$, such that the tree induced by the vertices in $A$ contains $u$ and the tree induced by the vertices in $B$ contains $v$. Denote these trees by $T_A$ and $T_B$. Then, $(T \setminus T_{uv}) \cup T_A \cup T_B$ is a tree cover of $G$ of size $T(G - e) + 1$. This shows that $T(G) - 1 \leq T(G - e)$.

We now show that $T(G - e) \leq T(G) + 1$. Suppose there is a minimum tree cover, $T$, of $G$ such that $u$ and $v$ are in separate trees. Then $T$ is a tree cover of $G - e$, so $T(G - e) \leq T(G)$. Otherwise, let $T$ be a minimum tree cover of $G$ that uses the edge $e$ (so $e$ is a simple edge by the definition of a tree cover), and let $T_e$ be the tree in $T$ that contains $e$. By deleting $e$ from $T_e$, we produce a tree cover of $G - e$ of size $T(G) + 1$. This shows that $T(G - e) \leq T(G) + 1$, which completes the proof. $\square$

The next theorem gives a bound that holds for the positive semidefinite maximum nullity of a graph, but the example that follows demonstrates that the analogous bound for the tree cover number fails.

A 2-separation of a graph $G = (V, E)$ is a pair of subgraphs $(G_1, G_2)$ such that $V(G_1) \cup V(G_2) = V, |V(G_1) \cap V(G_2)| = 2, E(G_1) \cup E(G_2) = E,$ and $E(G_1) \cap E(G_2) = \emptyset$.

**Theorem 4.** ([3], Theorem 2.8) Let $(G_1, G_2)$ be a 2-separation of a graph $G = (V, E)$, and let $H_1$ and $H_2$ be obtained from $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, by adding an edge between the vertices of $R = \{r_1, r_2\} \subset V_1 \cap V_2$. Then

$$M_+(G) = \max\{M_+(G_1) + M_+(G_2) - 2, M_+(H_1) + M_+(H_2) - 2\}$$

The analogous bound does not hold for the tree cover number. The next example provides a counterexample.
Example 5. For the graphs $G, G_1, G_2, H_1, H_2$ given in Figure 1 below, we have that $M_i(G_i) = 2, M_i(H_i) = 3$, and $T(G_i) = T(H_i) = 2$ for $i \in \{1, 2\}$. So by Theorem 4, $M_+(G) = 4$. However,

$$3 = T(G) > \max\{T(G_1) + T(G_2) - 2, T(H_1) + T(H_2) - 2\} = 2.$$

Figure 1: $G$ (top left), $G_1$ (top middle), $G_2$ (top right), $H_1$ (bottom left), $H_2$ (bottom right)
3 Characterizing Edges Required in a Minimum Tree Cover

Proposition 6. Let \( G = (V, E) \) be a graph such that \( uv \in E(G) \) is a bridge. Then \( uv \) is in a tree in every minimum tree cover of \( G \).

Proof. Note that there is no path from \( u \) to \( v \) that does not include \( uv \). Therefore, for any tree cover that does not include \( uv \), it must be the case that \( u \) and \( v \) are in separate trees. These two trees can be consolidated into one tree by adding the edge \( uv \).

We then ask the question: If an edge is required in every minimum tree cover, must it be a bridge? Figure 2 below shows that such an edge is not necessarily a bridge.

Example 7. Figure 2 below gives a graph whose tree cover number is 2. However, although \( uv \) is not a bridge, any tree cover that does not include \( uv \) is of size at least 3.

![Figure 2: for Example 7](image)

The next lemma gives us a way to determine if an edge is required in every minimum tree cover, given that we are able to compute the necessary tree cover numbers.

Theorem 8. Let \( G \) be a graph, \( u,v \in V(G) \), and \( uv \) is a simple edge in \( E(G) \). Let \( H \) be the graph obtained from \( G \) by adding a vertex such that \( V(H) = V(G) \cup \{w\} \) and \( E(H) = E(G) \cup \{uw,vw\} \), where \( uw \) and \( vw \) are simple edges. Then, \( uv \) is required in every minimum tree cover of \( G \) if and only if \( T(H) = T(G) + 1 \).

Proof. First observe that \( T(H) \leq T(G) + 1 \) since any tree cover of \( G \) together with \( \{w\} \) is a tree cover for \( H \). Let \( T = \{T_1, T_2, \ldots, T_k\} \) be a minimum tree cover of \( H \) such that \( w \in T_i \) for some \( i \). Since \( w,u \), and \( v \) cannot all be in the same tree, then either \( w \) is a leaf in \( T_i \) or \( T_i = \{w\} \). If \( w \) is a leaf in \( T_i \), then \( T_1, T_2, \ldots, T_i - w, T_{i+1}, \ldots, T_k \) is a tree cover of \( G \), so \( T(G) \leq T(H) \). If \( T_i = \{w\} \), then \( T \setminus T_i \) is a tree cover for \( G \), so \( T(G) \leq T(H) - 1 \). This shows that \( T(H) = T(G) \) or \( T(H) = T(G) + 1 \).

Suppose that \( uv \) is required in every minimum tree cover of \( G \). If \( w \) is a leaf in \( T_i \), then \( T_1, T_2, \ldots, T_i - w, T_{i+1}, \ldots, T_k \) is a tree cover of \( G \) with \( u \) and \( v \) in separate trees, so it follows that \( T(H) = T(G) + 1 \). If \( T_i = \{w\} \), then we also...
have that $T(H) = T(G) + 1$.
Suppose that there exists a minimum tree cover $T = \{T_1, T_2, \ldots, T_k\}$ of $G$ such that $u$ and $v$ are in different trees. If $u \in T_i$, we can create a tree cover of $H$ of size $k$ by adding the edge $uv$ to $E(T_i)$. In this case, $T(G) = T(H)$. 

One might think that if $H$ is a graph obtained from $G$ by adding the edge $uv$, and $uv$ is required in every minimum tree cover of $H$, then $T(G) = T(H) + 1$. However, this is not true. Example 9 provides a counterexample.

**Example 9.** It is easy to see that $T(G) = T(H) = 2$. (For $H$, take the set $\{1, u, v, 5\}$ and $\{2, 3, 4\}$ for example.) It can also be verified that $T(\hat{H}) = 3$. By Theorem 8, it follows that the edge $uv$ is required in every minimum tree cover of $H$.

![Figure 3: $G$, $H$ and $\hat{H}$ for Example 9](image-url)
4 Tree Cover Number of the Hypercube

The $d$-dimensional hypercube, denoted $Q_d$, is the simple graph with vertex set $\{0,1\}^d$ where two vertices are adjacent if and only if they differ in exactly one position. For example, the 2 dimensional hypercube is a square, shown in Figure 4 below, and the 3 dimensional hypercube is a cube, shown in Figure 5.

Equivalently, hypercubes can be inductively defined as the cartesian product of $d$ copies of the complete graph $K_2$. Hypercubes are a particular case of a larger family of graphs called Hamming graphs. The $d-$dimensional Hamming graph, denoted $H(d,q)$ is the graph with vertex set $\{0,\ldots,n-1\}^d$ where two vertices are adjacent if and only if they differ in exactly one position. Hamming graphs are of use in many areas including error-correcting codes, modeling heat diffusion, and association schemes in statistics. In this section, we show that the tree cover number of the $d$-dimensional hypercube is 2 for all $d \geq 2$.

Theorem 10. Let $Q_d$ be the $d-$dimensional hypercube graph. For all $d \geq 2$, $T(Q_d) = 2$.

Proof. For $d \in \{2,3,4,5\}$, the sets which induce a tree cover of size for $Q_d$ are explicitly listed in the appendix. Throughout this proof, the sets $\hat{T}_1$ and $\hat{T}_2$ are covers that will be used as preliminary steps to obtain the sets $T_1$ and $T_2$ that will induce a tree cover of size two for $Q_d$. The proof proceeds as follows: First we construct a cover and a tree cover of size two for $Q_6$. Using this tree cover, we construct a cover and a tree cover of size two for $Q_7$. We then inductively show that for $d \geq 8$ we can systematically construct a tree cover of size two using the covers and tree covers constructed for $Q_{d-1}$ and $Q_{d-2}$.

Consider the sets $\hat{T}_{16}$ and $\hat{T}_{26}$ given in the appendix. Note that $\{\hat{T}_{16}, \hat{T}_{26}\}$ is a cover for $Q_6$, and that $Q_6[\hat{T}_{16}]$ and $Q_6[\hat{T}_{17}]$ are both forests, each consisting of two disjoint trees. Let $x_{16} = (001101), x_{26} = (110010), y_{14} = (001001), y_{26} = (110100).$ Then $x_{16}$ and $x_{26}$ are in $\hat{T}_{16}$, and they are not in the same tree in $Q_6[\hat{T}_{16}]$. Similarly, $y_{14}$ and $y_{26}$ are in $\hat{T}_{26}$, and they are not in the same tree in $Q_6[\hat{T}_{26}]$. By swapping $x_{14}$ and $y_{14}$, the resulting sets $T_{16}$ and $T_{26}$ (listed in the appendix) induce a tree cover for $Q_6$ of size two.

To obtain a tree cover of size two for $Q_7$, we begin by adding a 0 to the beginning of each element in $T_{16}$, and a 1 to the beginning of each element in $T_{26}$.
Denote these sets by $T_{1,0}$ and $T_{2,1}$, respectively, and let $\hat{T}_{1,0} := T_{1,0} \cup T_{2,1}$.
Similarly, we construct the sets $T_{1,0}$ and $T_{2,0}$, and let $\hat{T}_{2,0} := T_{1,0} \cup T_{2,0}$ (see appendix). Then, both $Q_1[\hat{T}_{1,0}]$ and $Q_1[\hat{T}_{2,0}]$ are forests consisting of two disjoint trees. By swapping $0x_{2_0}$ and $0y_{2_0}$, the resulting sets $\hat{T}_{1,0}$ and $\hat{T}_{2,0}$ (given in appendix) induce a tree cover of size two for $Q_1$.

We proceed by induction to prove the claim for $Q_d$ with $d \geq 8$. Suppose that we have constructed the sets $\hat{T}_{1,d-2} = \{x_1, x_2, \ldots, x_n\}$ and $\hat{T}_{2,d-2} = \{y_1, y_2, \ldots, y_n\}$ such that $\{\hat{T}_{1,d-2}, \hat{T}_{2,d-2}\}$ gives a cover for $Q_{d-2}$ satisfying the following conditions:

1. $Q_{d-2}[\hat{T}_{1,d-2}]$ and $Q_{d-2}[\hat{T}_{2,d-2}]$ are forest composed of two disjoint trees.
2. Swapping $x_1$ and $y_1$ results in sets
   
   $T_{1,d-2} = \{y_1, x_2, \ldots, x_n\}$
   $T_{2,d-2} = \{x_1, y_2, \ldots, y_n\}$

   that induce a tree cover of $Q_{d-2}$ of size two.
3. For the cover
   
   $\hat{T}_{1,d-1} = T_{1,d-2,0} \cup T_{2,d-2,1} = \{0y_1, 0x_2, 0x_3, \ldots, 0x_n, 1x_1, 1y_2, \ldots, 1y_n\}$
   $\hat{T}_{2,d-1} = T_{2,d-2,0} \cup T_{1,d-2,1} = \{0x_1, 0y_2, 0y_3, \ldots, 0y_n, 1y_1, 1x_2, \ldots, 1x_n\}$

   of $Q_{d-1}$, swapping $0x_2 \in \hat{T}_{1,d-1}$ and $0y_2 \in \hat{T}_{2,d-1}$ results in sets
   
   $T_{1,d-1} = \{0y_1, 0y_2, 0x_3, \ldots, 0x_n, 1x_1, 1y_2, \ldots, 1y_n\}$
   $T_{2,d-1} = \{0x_1, 0x_2, 0y_3, \ldots, 0y_n, 1y_1, 1x_2, \ldots, 1x_n\}$

   for $Q_{d-1}$ that induced a tree cover of $Q_{d-1}$ of size two.
4. $x_1$ and $x_2$ are not in the same induced tree in $\hat{T}_{1,d-2}$
5. $y_1$ and $y_2$ are not in the same induced tree in $\hat{T}_{2,d-2}$

(We are also assuming that $x_i \neq x_j, y_i \neq y_j$ for $i \neq j$, and $x_i \neq y_j$ for all $i, j$).

Then we can construct a cover for $Q_d$ such that swapping two of the elements in the cover will result in a tree cover of size two for $Q_d$. Furthermore, we show that the constructed cover and tree cover for $Q_d$, together with the constructed cover and tree cover for $Q_{d-1}$, still satisfy the above hypotheses, which proves the claim for all $d \geq 8$.

We first construct a cover $\{\hat{T}_{1,d}, \hat{T}_{2,d}\}$ for $Q_d$ in the following way:

$\hat{T}_{1,d} = T_{1,d-1,0} \cup T_{2,d-1,1} = \{00y_1, 00y_2, 00x_3, \ldots, 00x_n, 01x_1, 01y_2, 01y_3, \ldots, 01y_n, 10x_2, 10y_3, 10y_4, \ldots, 10y_n, 11y_1, 11x_2, 11x_3, \ldots, 11x_n\}$

$\hat{T}_{2,d} = T_{2,d-1,0} \cup T_{1,d-1,1} = \{00x_1, 00x_2, 00y_3, \ldots, 00y_n, 01y_1, 01x_2, 01x_3, \ldots, 01x_n, 10y_1, 10y_2, 10x_3, \ldots, 10x_n, 11x_1, 11y_2, 11y_3, \ldots, 11y_n\}$. 

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By definition $Q_{d-1}[T_{d-1}]$ and $Q_{d-1}[T_{d-1}]$ are two disjoint trees, it follows that $Q_d[T_{d}]$ is a forest consisting of two disjoint trees. Similarly, $Q_d[T_{2d}]$ is a forest consisting of two disjoint trees. By swapping $01x_1$ and $01y_1$, we obtain the sets

\[
T_{1_d} = \{00y_1,00y_2,00x_3,\ldots,00x_n,01y_1,01y_2,01y_3,\ldots,01y_n, \\
10x_1,10x_2,10y_3,\ldots,10y_6,11y_1,11x_2,11x_3,\ldots,11x_n\}
\]

\[
T_{2_d} = \{00x_1,00x_2,00y_3,\ldots,00y_n,01x_1,01x_2,01x_3,\ldots,01x_n, \\
10y_1,10y_2,10x_3,\ldots,10x_6,11x_1,11y_2,11y_3,\ldots,11y_n\}.
\]

We now show that \{\(Q_d[T_{1_d}], Q_d[T_{2d}]\)\} is a tree cover for \(Q_d\) of size two by showing:

1. \(Q_d[T_{1_d}]\) and \(Q_d[T_{2d}]\) are forests (i.e., there are no cycles in each of \(Q_d[T_{1d}]\) and \(Q_d[T_{2d}]\)).

2. Both \(Q_d[T_{1d}]\) and \(Q_d[T_{2d}]\) are connected graphs.

We show that \(Q_d[T_{1d}]\) is a forest (a similar argument shows that \(Q_d[T_{2d}]\) is a forest). From our construction \(Q_d[\hat{T}_{d}]\) is a forest composed of 2 trees, \(Q_d[A]\) and \(Q_d[B]\) where

\[
\hat{A} := \{00y_1,00y_2,00x_3,\ldots,00x_n,01x_1,01y_2,01y_3,\ldots,01y_n\}
\]

\[
B := \{10x_1,10x_2,10y_3,\ldots,10y_n,11y_1,11x_2,11x_3,\ldots,11x_n\}.
\]

By definition \(T_{1d} = (\hat{T}_{1d} \setminus \{01x_1\}) \cup \{01y_1\}\). By removing \(01x_1\) from \(\hat{T}_{1d}\), \(B\) is not affected, and \(Q_d[\hat{A}\setminus\{01x_1\}]\) is now the union of \(\text{deg}(01x_1)\) disjoint trees. We now show that by adding \(01y_1\) to \(\hat{T}_{1d}\) \(\setminus\{01x_1\}\), no cycles are created in \(Q_d[T_{1d}]\).

Define \(A = \{00y_1,00y_2,00x_3,\ldots,00x_n,01y_1,01y_2,01y_3,\ldots,01y_n\}\) (note that \(T_{1d} = A \cup B\)). Between \(A\) and \(B\), the only vertices that are adjacent are \(01y_1\) and \(11y_1\) (everything else differs in more than one position). Hence, if there is a cycle in \(Q_d[T_{1d}]\), it must be in \(Q_d[A]\). Since \(Q_d[A \setminus \{01y_1\}] = Q_d[\hat{A}\setminus\{01x_1\}]\) is a forest composed of \(\text{deg}(01x_1)\) trees, if there is a cycle in \(Q_d[A]\) it must involve \(01y_1\). We will now show that it is not possible to have a cycle involving \(01y_1\), hence no cycle is possible in \(Q_d[T_{1d}]\).

Note that there is an edge between \(00y_1\) and \(01y_1\), and that there are no edges between \(01y_1\) and any of \(00y_2,00x_3,\ldots,00x_n\). Thus, the neighbors of \(01y_1\) in \(Q_d[A]\) are \(00y_1\) and a subset of \(\{01y_3,01y_4,\ldots,01y_n\}\) (since \(y_1\) is not adjacent to \(y_2\) by condition (5) above, then \(01y_1\) is not adjacent to \(01y_2\)). Let \(01y_i\) and \(01y_j\), \(i \neq j\), be arbitrary neighbors of \(01y_1\). We show that:

(a) There is no path from \(01y_i\) to \(01y_j\) in \(Q_d[A]\) for \(i,j \in \{3,4,\ldots,n\}\) that does not include \(01y_1\).

(b) There is no path from \(00y_1\) to \(01y_i\) in \(Q_d[A]\) that does not include \(01y_1\).
To see (a), note that from condition (1), \( Q_{d-2}[\{y_1, \ldots, y_n\}] \) is a forest of two disjoint trees. This implies that \( Q_d[\{01y_1, \ldots, 01y_n\}] \) is a forest of two disjoint trees. Then, within \( Q_d[\{01y_1, \ldots, 01y_n\}] \) there is no path from 01\( y_i \) to 01\( y_j \) that does not include 01\( y_1 \). Note that vertices of \( \{01y_1, 01y_2, \ldots, 01y_n\} \) are not adjacent to any vertices in \( A \) except for possibly each other and 01\( y_1 \) and 01\( y_2 \). Thus, any path from 01\( y_i \) to 01\( y_j \), not including 01\( y_1 \), must include 01\( y_2 \). By condition (1), 01\( y_1 \) and 01\( y_2 \) are not in the same induced tree of \( Q_{d-2}[\{y_1, \ldots, y_n\}] \), so 01\( y_1 \) and 01\( y_2 \) are not in the same induced tree of \( Q_d[\{01y_1, \ldots, 01y_n\}] \). Since 01\( y_1 \) and 01\( y_j \) are neighbors of 01\( y_1 \), and 01\( y_1 \) is not in the same induced tree as 01\( y_2 \) in \( Q_d[\{01y_1, \ldots, 01y_n\}] \), then 01\( y_1 \) and 01\( y_j \) are not in the same induced tree as 01\( y_2 \). Thus, the only path from 01\( y_i \) to 01\( y_j \) is \( (01y_1, 01y_1, 01y_j) \).

For (b), we have that the vertices in the set \( \{01y_1, 01y_2, \ldots, 01y_n\} \) are not connected in \( Q_d[A] \) to any vertices in \( A \) except for possibly each other and 01\( y_1 \). We also have that 01\( y_1 \) is not adjacent to 00\( y_i \) in \( Q_d[A] \). So any path from 01\( y_i \) to 00\( y_j \) must include 01\( y_1 \).

Next we show that \( Q_d[T_{1d}] \) is connected (a similar argument shows that \( Q_d[T_{2d}] \) is connected). Recall from the hypotheses that \( Q_{d-2}[\hat{T}_{2d-2}] = Q_{d-2}[\{y_1, y_2, \ldots, y_n\}] \) is a forest consisting of two disjoint trees, and \( Q_{d-2}[\hat{T}_{2d-2}] = Q_{d-2}[\{x_1, y_2, \ldots, y_n\}] \) is a tree. This implies that 01\( y_1 \) has exactly one fewer neighbor among \( y_2, \ldots, y_n \) than 01\( x_1 \). To see this, note that \( Q_{d-2}[\hat{T}_{2d-2} \setminus \{y_1\}] \) is composed of \( 1 + \deg(01y_1) \) trees. Since \( Q_{d-2}[\hat{T}_{2d-2}] = Q_{d-2}[\hat{T}_{2d-2} \setminus \{y_1\} \cup \{x_1\}] \) is a tree, we must have \( \deg(01x_1) = 1 + \deg(01y_1) \). Therefore, 01\( y_1 \) must have one less neighbor than 01\( x_1 \) among 01\( y_2, \ldots, 01y_n \). Hence, 01\( y_1 \) and 01\( x_1 \) have the same number of neighbors in \( A \), and thus 01\( y_1 \) has one more neighbor than 01\( x_1 \) in \( T_{1d} \). We will now show that this last statement implies that \( Q_d[T_{1d}] \) is connected.

Since the graphs induced by \( T_{1d-1} \) and \( T_{2d-1} \) are trees, then \( Q_d[T_{1d}] = Q_d[T_{1d-1} \cup T_{2d-1}] \) is a forest consisting of two disjoint trees. Hence, \( Q_d[T_{1d} \setminus \{01x_1\}] \) is a forest consisting of \( 1 + \deg(01x_1) \) trees. Since \( \deg(01y_1) = 1 + \deg(01x_1) \), and since \( Q_d[T_{1d}] \) has no cycles, we have that each of the edges of 01\( y_1 \) must be connected to a different component of the forest. Therefore, \( Q_d[T_{1d}] \) is a tree. An analogous argument shows that \( Q_d[T_{2d}] \) is a tree. Thus, \( \{Q_d[T_{1d}], Q_d[T_{2d}]\} \) is a tree cover of size two of \( Q_d \).

We now show that the covers and tree covers constructed for \( Q_{d-1} \) and \( Q_d \) satisfy the induction hypotheses. Note that since \( Q_{d-2}[T_{1d-2}] \) and \( Q_{d-2}[T_{2d-2}] \) are two disjoint trees, it follows from construction that \( Q_d[T_{1d-1}] \) is a forest consisting of two disjoint trees. Similarly, \( Q_d[T_{2d-1}] \) is a forest consisting of two disjoint trees, satisfying condition (1). For clarity, we relabel the vertices of \( T_{1d-1} \) and \( T_{2d-1} \) such that \( T_{1d-1} = \{w_1, \ldots, w_m\} \) and \( T_{2d-1} = \{z_1, \ldots, z_n\} \), where \( w_1 = 0x_2, w_2 = 1x_1, z_1 = 0y_2, z_2 = 1y_1 \). Then by condition (3), swapping \( w_1 \) and \( z_1 \) results in sets \( T_{1d-1} = \{z_1, w_2, \ldots, w_m\} \) and \( T_{2d-1} = \{w_1, z_2, \ldots, z_n\} \).
that induce a tree cover of \(Q_{d-1}\) of size two, which shows that condition (2) is satisfied. Note that with this relabeling, the sets \(\hat{T}_1d\) and \(\hat{T}_2d\) become

\[
\hat{T}_1d = T_{1d-1,0} \cup T_{2d-1,1} = \{0z_1, 0w_2, 0w_3, \ldots, 0w_m, 1w_1, 1z, \ldots, 1z_m\}
\]

\[
\hat{T}_2d = T_{2d-1,0} \cup T_{1d-1,1} = \{0w_1, 0z_2, 0y_3, \ldots, 0z_m, 1z_1, 1w_2, \ldots, 1w_m\},
\]

and we have shown above that swapping \(0w_2 = 01x_1\) and \(0z_2 = 01y_1\) results in the sets \(T_1d\) and \(T_2d\) which induce a tree cover of size two for \(Q_d\), satisfying condition (3). Furthermore, since \(w_1 = 0x_2 \in T_{1d-2,0}\) and \(w_2 = 1x_1 \in T_{2d-2,1}\), we have that \(w_1\) and \(w_2\) are not in the same induced tree in \(Q_{d-1}[\hat{T}_{1d-1}]\). Similarly, \(z_1 = 0y_2 \in T_{2d-2,0}\) and \(z_2 = 1y_1 \in T_{1d-2,1}\), so \(z_1\) and \(z_2\) are not in the same induced tree in \(Q_{d-1}[\hat{T}_{2d-1}]\), showing that conditions (4) and (5) are satisfied.

Since the hypotheses still hold with the constructed covers and tree covers of \(Q_{d-1}\) and \(Q_d\), then it follows, by inductively applying the above argument, that \(T(Q_d) = 2\) for all \(d\).

One may wonder why the base case of the proof starts with \(Q_6\) and \(Q_7\). We would like to note that starting as early as \(d = 2\), we were able to use a tree cover of \(Q_d\) to produce a cover for \(Q_{d+1}\) such that there exists two vertices that could be swapped in order to produce a tree cover for \(Q_{d+1}\). In fact, this is how we constructed the tree covers for \(Q_3, Q_4, Q_5\) given in the appendix. However, there is a choice to be made when switching vertices, and the point at which the above constructive pattern holds is dependent upon the initial choice of vertices that are swapped. For example, we experimented with using a different initial swap and found that the pattern did not hold until \(d = 11\) or later. It may also be the case that there is an initial swap that allows the pattern to begin sooner than \(d = 8\). This is a very interesting phenomenon that is worth further exploration.

We also investigated the idea of generalizing the above proof to all Hamming graphs. For \(H(2, 3)\), we found that \(T(H(2, 3)) = 3\), and evidence suggests that \(T(H(d, q)) = q\).

**Conjecture 11.** Let \(H(d, q)\) be the Hamming graph of dimension \(d\). Then, \(T(H(d, q)) = q\).

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5 Appendix

5.1 Sets which induce tree covers for $Q_d$ for $d \in \{2, 3, 4, 5\}$

For $d = 2$:
\[ T_{12} = \{(00), (01)\}, \]
\[ T_{22} = \{(11), (10)\}. \]

For $d = 3$:
\[ T_{13} = \{(010), (000), (001), (110)\}, \]
\[ T_{23} = \{(111), (011), (100), (101)\}. \]

For $d = 4$:
\[ T_{14} = \{(0011), (0010), (0000), (0110), (1111), (1011), (1100), (1101)\}, \]
\[ T_{24} = \{(0001), (0111), (0100), (0101), (1010), (1000), (1001), (1110)\}. \]

For $d = 5$
\[ T_{15} = \{(000100), (000111), (000000), (000110), (001011), (001100), (011001), \]
\[ \quad (100011), (101101), (110101), (110110), (111000), (111011), (111010), (111100), (111101), (111111)\}, \]
\[ T_{25} = \{(000010), (000001), (001111), (010111), (010000), (010010), (100111), (111000), (110110), (111111)\}. \]

5.2 Sets Used in Proof of Theorem 10

\[ \hat{T}_{16} = \{(000000), (000011), (000100), (000110), (001011), (001100), (001111), \]
\[ \quad (010001), (010100), (010111), (011011), (011100), (011101), (011110), \]
\[ \quad (100001), (100010), (100110), (110011), (110100), (111001), (111010), (111100), (111101), (111111)\}. \]

\[ \hat{T}_{26} = \{(000001), (000010), (000101), (000111), (001000), (001010), (001100), \]
\[ \quad (010010), (010110), (011011), (011101), (011110), (100000), (100011), (100100), (100101), \]
\[ \quad (101011), (110001), (110010), (110100), (110110), (111000), (111001), (111010), (111110)\}. \]

\[ T_{16} = \{(001001), (000000), (000011), (000100), (000110), (001011), (001100), (001111), \]
\[ \quad (010001), (010100), (010111), (011011), (011100), (011101), (011111), (100001), (100010), \]
\[ \quad (100100), (100111), (110000), (110001), (110010), (110011), (110100), (110101), \]
\[ \quad (111000), (111001), (111010), (111011), (111100), (111101), (111111)\}. \]
\[ T_{2c} = \{(001101), (000001), (000010), (000101), (000111), (001000), (001010), (001110), (010000), (010010), (010011), (010110), (011100), (011101), (011111), (100000), (100011), (100100), (100110), (101011), (101110), (110001), (110100), (110111), (111000), (111001), (111100), (111101), (111110)\} \]

\[ \hat{T}_{17} = \{(0000000), (0000011), (0000100), (0000101), (0000111), (0001001), (0010001), (0010111), (0011000), (0011010), (0011100), (0100001), (0100101), (0100111), (0101000), (0101010), (0101011), (0101100), (0101101), (0101110), (0110000), (0110011), (0110100), (0110101), (0110110), (0111000), (0111001), (0111010), (0111011), (0111100), (0111101), (0111110), (0111111)\} \]

\[ \hat{T}_{27} = \{(0000001), (0000010), (0000011), (0000100), (0000101), (0000110), (0001000), (0001001), (0001011), (0010000), (0010100), (0010101), (0010111), (0011000), (0011010), (0011011), (0011100), (0011101), (0011111), (0100000), (0100010), (0100011), (0100100), (0100101), (0100110), (0101000), (0101010), (0101011), (0101100), (0101101), (0101110), (0110000), (0110001), (0110010), (0110011), (0110100), (0110101), (0110110), (0111000), (0111001), (0111010), (0111011), (0111100), (0111101), (0111110), (0111111)\} \]

\[ T_{17} = \{(0000000), (0000001), (0000010), (0000011), (0000101), (0000110), (0001000), (0001001), (0001011), (0010000), (0010010), (0010011), (0010100), (0010101), (0010110), (0011000), (0011010), (0011011), (0011100), (0011101), (0011110), (0011111), (0100000), (0100001), (0100010), (0100011), (0100101), (0100110), (0101000), (0101001), (0101011), (0101100), (0101101), (0101110), (0110000), (0110001), (0110010), (0110011), (0110101), (0110110), (0111000), (0111001), (0111010), (0111011), (0111100), (0111101), (0111110), (0111111)\} \]
$$T_{2^7} = \{(0000000), (0000010), (0000101), (0000110), (0001010), (0001101), (0001110), (0010000), (0010011), (0010100), (0010110), (0011000), (0011011), (0011011), (0011100), (0011100), (0011111), (0100000), (0100011), (0100100), (0100110), (0101000), (0101011), (0101100), (0101100), (0101111), (0110001), (0110010), (0110101), (0110111), (0111000), (0111000), (0111011), (0111100), (0111100), (0111111), (1000001), (1000010), (1000101), (1000111), (1001001), (1001010), (1001101), (1001110), (1010000), (1010011), (1010100), (1010111), (1011000), (1011000), (1011011), (1011100), (1011100), (1011111), (1100001), (1100010), (1100101), (1100111), (1101000), (1101000), (1101011), (1101101), (1101101), (1101111), (1110001), (1110010), (1110101), (1110111), (1111000), (1111000), (1111011), (1111100), (1111010), (1111101) \}.$$ 

References


